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Comments about quantum symmetries of $SU(3)$ graphs

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Abstract

For the $SU(3)$ system of graphs generalizing the ADE Dynkin digrams in the classification of modular invariant partition functions in CFT, we present a general collection of algebraic objects and relations that describe fusion properties and quantum symmetries associated with the corresponding Ocneanu quantum groupoïds. We also summarize the properties of the individual members of this system.

Keywords: Conformal field theory; Modular invariance; Coxeter–Dynkin graphs; Fusion algebra; Quantum symmetries; Quantum groupoïds

1. Introduction

The stage

Over the last fifteen years or so, investigations performed in a number of research fields belonging to theoretical physics or to mathematics have suggested the existence of "fundamental objects" generalizing the usual simply laced ADE Dynkin diagrams. Let us mention a few of these fields: Statistical mechanics, string theory, quantum gravity, conformal field theory, theory of bimodules, Von Neumann algebras, sector theory, (weak) Hopf algebras, modular categories.

Properties of the algebraic structures associated with the choice of such a fundamental object have been analysed independently by several groups of people, with their own tools and terminology. The results obtained by these different schools are not always easy to compare, or even to comprehend, because of the required background and specificity of the language.

However, at the heart of any such fundamental object we meet a graph (or the adjacency matrix that encodes this graph). We believe that many important and useful results can be described in an elementary way obtained from the combinatorial data provided by the graph itself, or from some kind of attached modular data [23].

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Roughly speaking, if we have a modular invariant (but not any kind of modular invariant), we have a (particular type of) quantum groupoïd, and conversely. Now every such quantum groupoïd is encoded by a graph, and this graph leads naturally to two (in general distinct) character theories: One is the so-called fusion algebra, and the other is the algebra of quantum symmetries. This is the story that we want to tell. But we want to tell it in simple words, using elementary mathematics. And we want to tell it in the case of the $SU(3)$ system of graphs, i.e., the so-called "Di Francesco–Zuber diagrams" that generalize the familiar ADE Dynkin diagrams.

As already mentioned, several groups of people (without trying to be exhaustive, we can cite [6,42,43,23,20,45,3,12,15,8]) have investigated related topics over recent years. We believe that only Ocneanu has actually worked out all these examples in detail, with his own language, from the point of view of the study of quantum symmetries, but his results are unfortunately unpublished and not available.

Purpose

The purpose of this article is threefold.

(1) To present, in a synthetic and elementary way, a collection of algebraic objects describing fusion properties and quantum symmetries associated with graphs belonging to (higher) Coxeter–Dynkin systems.
(2) To present a summary of results concerning members of the $SU(3)$ system.
(3) To make a number of comments about the various aspects of this subject, and, in some cases, to establish a distinction between what is known and what is believed to be true.

Warning

This paper is not a review. If it is true that many results recalled here can be found in the literature, maybe with another language or perspective, many others cannot be found elsewhere. It may well be that a number of these results have been privately worked out by several people, but, if so, they are not available. What we present here, including a good part of the terminology itself, is mostly the result of our own understanding, that has been growing over the years.

However, this paper is not a detailed research paper either. Indeed, it is, in a sense, too short. Every single example summarized in Section 6, for instance, gives rise to interesting, and sometimes difficult, problems, and would certainly be worth a dedicated article. What we put in this section is only what we think should be remembered once all the details have been forgotten. This, admittedly, is a partial viewpoint.

We want this paper to be used as a compendium of results, terminology, and remarks.

Plan

The plan of this article is as follows. In the next section we summarize the properties of the $\mathcal{A}$ system, i.e., the Weyl alcoves at level k, from the viewpoint of fusion and graph algebras. In Section 3, we describe general properties associated with any member of the $SU(3)$ system of graphs. This applies, in particular, to the $\mathcal{A}$ graphs themselves, but they are very particular, and this is why we singled them out. In the fourth section, we describe, in plain terms, the Ocneanu quantum groupoïd associated with a graph G, or, better, with a pair $(G, \mathcal{A}_k)$. We do not give however any information about the methods that allow one to compute the values of the corresponding cells; this is a most essential question but it should be dealt with in another publication. In the fifth section we describe the equations that allow one to recover the algebra of quantum symmetries (and sometimes the graph itself) from the data provided by a modular invariant, the leitmotiv of this section being the so-called "modular splitting technique". Although we have repeatedly used this technique to solve several quite involved examples briefly described in Section 6, we do not explicitly discuss here our method of resolution but refer to forthcoming articles (or theses) for these — important — details [27,26,24]. In Section 6 we summarize what is known, or at least what we know, about the structure of the algebra of quantum symmetries for each member of the $SU(3)$ series. At this point we should stress that the graphs themselves, together with their fusion properties (relations with the $\mathcal{A}$ system) or with the associated modular invariants, were discovered and described long ago (by Di Francesco and Zuber [17]). Several aspects related to the theory of sectors, or to the theory of bimodules have also been investigated independently by different groups of people [2,3,20,18]. However we believe that only Ocneanu performed a detailed analysis of the algebra of quantum symmetries associated with all these diagrams and three of us remember vividly the poster describing the Cayley graph for the generators of the algebra that we call $Oc(\mathcal{E}_9)$, on one of the walls of the Bariloche conference lecture hall, during the January 2000 summer (!) school. However, this material was never published or even made public on the internet. Our techniques may be sometimes clumsy but we hope that they are understandable and will draw the attention of potential readers to this fascinating subject. We now return to the plan of our paper and mention that the last section (the 7th) is devoted to a set of final remarks describing possible new directions or open problems.

2. $\mathcal{A}_k$ graphs

2.1. First properties

The $\mathcal{A}_k$ graphs are obtained as truncations of the Weyl chambers of $SU(N)$ at some level (Weyl alcoves). They have a level k and a (generalized) Coxeter number $\kappa = k + N$. From now on $N = 3$.

Vertices

Vertices λ may be labelled by Dynkin labels (λ_1, λ_2), with $0 \leq \lambda_1 + \lambda_2 \leq k$, by shifted Dynkin labels $\{\lambda_1 + 1, \lambda_2 + 1\} = (\lambda_1, \lambda_2)$, or by Young tableaux[1] $Y[p, q]$, $p = \lambda_1 + \lambda_2$, $q = \lambda_2$. For instance, the unit vertex (trivial representation) is $(0, 0) = \{1, 1\} = Y[0, 0]$, the fundamental vertex $(1, 0) = \{2, 1\} = Y[1, 0]$ and its conjugate $(0, 1) = \{1, 2\} = Y[1, 1]$. The graph $\mathcal{A}_k$ possesses $d_{\mathcal{A}_k} = (k + 1)(k + 2)/2$ vertices. The vector space spanned by these vertices is also called $\mathcal{A}_k$.

Conjugation

The graph $\mathcal{A}_k$ has an involution $\star$: $(\lambda_1, \lambda_2) \to (\lambda_2, \lambda_1)$ called conjugation.

Triality

Each vertex λ possesses a triality $t(\lambda) = \lambda_1 - \lambda_2 \bmod 3$. It is equal to the number of boxes modulo 3 of the corresponding Young tableau. Conjugation leaves triality 0 invariant and interchanges 1 and 2.

Edges

Edges are oriented. They only connect vertices of increasing triality, by step +1, i.e. we choose one of the two possible adjacency matrices (the other is its transpose, with the edges in the opposite direction).

2.2. Spectral properties

Exponents and norm

The adjacency matrix of the graph $\mathcal{A}_k$ possesses $d_{\mathcal{A}_k}$ distinct complex eigenvalues [50]:

$$\beta(r_1, r_2) = \mathrm{e}^{-\frac{2i\pi(2(r_1+1)+(r_2+1))}{3\kappa}} \left(1 + \mathrm{e}^{\frac{2i\pi(r_1+1)}{\kappa}} + \mathrm{e}^{\frac{2i\pi((r_1+1)+(r_2+1))}{\kappa}}\right), \tag{1}$$

where $r_1, r_2 \geq 0$ and $r_1 + r_2 \leq k$. Such pairs of integers (r_1, r_2) are called exponents of the graph $\mathcal{A}_k$. The vertices of the $\mathcal{A}_k$ graph can be indexed by the same set of integer pairs (r_1, r_2): They coincide with the Dynkin labels (λ_1, λ_2). The set of eigenvalues is invariant under the group $\mathbb{Z}_3$. One of these eigenvalues $\beta \doteq \beta(0, 0)$ is real, positive, and of largest absolute value. It is called the norm of the graph, and it is equal to $\beta = 1 + 2\cos(2\pi/\kappa)$.

Class vectors, dimension vector and quantum dimensions

Normalized eigenvectors of the adjacency matrix are denoted as c_{r_1,r_2}. They can be called "class vectors" in analogy with the situation that prevails for finite groups. Here "normalized" means that the first component[2] of each class vector, corresponding to the unit vertex, is set to 1. The normalized eigenvector associated with the biggest eigenvalue β is called the dimension vector, or the Perron–Frobenius vector. Its components define the quantum dimensions of the corresponding vertices of $\mathcal{A}_k$. The quantum dimension of a given vertex $\lambda = (\lambda_1, \lambda_2)$ is given by the q-analog of the classical formula for dimensions of $SU(3)$ irreps, usual numbers being replaced by quantum numbers: $q\dim(\lambda) = (1/[2]_q)([\lambda_1 + 1]_q[\lambda_2 + 1]_q[\lambda_1 + \lambda_2 + 2]_q)$, where $q = \exp(i\pi/\kappa)$ is a root of unity and $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. The norm β itself is equal to the quantum dimension of the fundamental vertices (1, 0) and (0, 1). The sum of squares of the quantum dimensions is called the order or the quantum mass of $\mathcal{A}_k$ and denoted as $m(\mathcal{A}_k)$.

[1] p (resp. q) is the number of boxes in the first (resp. second) line.

[2] We assume that an order has been chosen on the set of vertices and that the unit vertex comes first.

2.3. Fusion algebra

The vector space $\mathcal{A}_k$ possesses an associative (and commutative) algebra structure: It is an algebra with unity, vertex (0, 0), and two generators, vertices (1, 0) and (0, 1), called “fundamental generators”. The graph of multiplication by the first generator (1, 0) is encoded by the (oriented) graph $\mathcal{A}_k$: The product of a given vertex λ by the fundamental (1, 0) is given by the sum of vertices μ such that there is an edge going from λ to μ on the graph. Equivalently, this multiplication is encoded by the adjacency matrix $N_{(1,0)}$ of the graph. Multiplication by the other fundamental generator is obtained by reversing the arrows.

Fusion matrices

Multiplication by generators $\lambda = (\lambda_1, \lambda_2)$ is described by matrices N_λ, called fusion matrices. The identity is $N_{(0,0)} = \mathbb{1}_{d_{\mathcal{A}}}$. The other fusion matrices are obtained, once $N_{(1,0)}$ is known, from the known recurrence relation for coupling of irreducible $SU(3)$ representations (that we — of course — truncate at level k):

$$\begin{aligned} &N_{(\lambda,\mu)} = N_{(1,0)}\, N_{(\lambda-1,\mu)} - N_{(\lambda-1,\mu-1)} - N_{(\lambda-2,\mu+1)} \qquad \text{if } \mu \neq 0 \\ &N_{(\lambda,0)} = N_{(1,0)}\, N_{(\lambda-1,0)} - N_{(\lambda-2,1)} \\ &N_{(0,\lambda)} = (N_{(\lambda,0)})^{tr} \end{aligned} \tag{2}$$

where matrices $N_{(\lambda,\mu)} = 0$ if $\lambda = -1$ or $k+1$ or if $\mu = -1$ or $k+1$, and are periodic in the (λ, μ) plane — the periodicity cell is a Weyl alcove and there are six of them around the origin $\{1, 1\} = (0, 0)$. These matrices have non negative integer entries $(N_\lambda)_{\mu\nu} = N_{\lambda\mu}^{\nu}$ called fusion coefficients. They form a faithful representation of the fusion algebra:

$$N_\lambda N_\mu = \sum_\nu N_{\lambda\mu}^{\nu}\, N_\nu. \tag{3}$$

Conjugation (operation $\star$) on these matrices is obtained by transposition.

Essential paths (also called horizontal paths)

Since fusion matrices N_λ have non negative integer entries, one can associate a graph with every fusion matrix. If the matrix element $(N_\lambda)_{\mu\nu} = p$, we introduce p oriented edges from the vertex μ to the vertex ν. Such an edge is called an essential path of type λ from μ to ν. Remember that these indices are themselves Young tableaux. The graph associated with the fundamental generator (1, 0) is the $\mathcal{A}_k$ graph itself.

2.4. Modular considerations

The graphs $\mathcal{A}_k$ support a representation of the group $SL(2, Z)$. This group is generated by two transformations S and T satisfying $S^2 = (ST)^3 = C$, with $C^2 = 1$. The modular group itself, called $PSL(2, Z)$ is the quotient of this group by the relation $C = 1$.

The modular generator S

The adjacency matrix of $\mathcal{A}_k$ can be diagonalized by a matrix constructed from the set of eigenvectors (all eigenvalues are distinct). As fusion matrices N_λ commute, this matrix therefore diagonalizes all fusion matrices. Each line of this matrix is given by a (renormalized) class vector. We renormalize the lines in order that each line is of norm 1. We therefore divide each class vector by its norm. The diagonalizing matrix obtained is then unitary but not a priori symmetric, and not necessarily related to the generator of the modular group. To write such a unitarizing matrix, one has first to choose an order on the set of eigenvalues (this fixes the ordering of line vectors), and also an order on the set of vertices of the graph (this fixes the ordering of the components for each line). One member of this family of unitarizing matrices gives the modular generator S. The point is that vertices of the graph $\mathcal{A}_k$ have to be indexed by the same set of integers as the eigenvalues themselves.[3] So, whatever the order we choose on the set of vertices, we decide to choose the same order on the set of eigenvalues. This procedure determines — for each ordering of the

[3] We thank O. Ogievetsky for this remark.

vertices — a particular unitarizing matrix which can be identified with the modular generator S. It coincides with the expression explicitly given by the formula [28,22]:

$$\begin{aligned}S_{\lambda\mu} = \frac{-\mathrm{i}}{\sqrt{3\kappa}}(&e_\kappa[2\lambda_1\mu_1+\lambda_1\mu_2+\lambda_2\mu_1+2\lambda_2\mu_2]-e_\kappa[-\lambda_1\mu_1+\lambda_1\mu_2+\lambda_2\mu_1+2\lambda_2\mu_2]\\ &-e_\kappa[2\lambda_1\mu_1+\lambda_1\mu_2+\lambda_2\mu_1-\lambda_2\mu_2]+e_\kappa[-\lambda_1\mu_1+\lambda_1\mu_2-2\lambda_2\mu_1-\lambda_2\mu_2]\\ &+e_\kappa[-\lambda_1\mu_1-2\lambda_1\mu_2+\lambda_2\mu_1-\lambda_2\mu_2]-e_\kappa[-\lambda_1\mu_1-2\lambda_1\mu_2-2\lambda_2\mu_1-\lambda_2\mu_2]),\end{aligned}$$

where $e_\kappa[x] := \exp[\frac{-2\mathrm{i}\pi x}{3\kappa}]$ and where the vertices are labelled by shifted Dynkin labels $\lambda = \{\lambda_1, \lambda_2\}$, $\mu = \{\mu_1, \mu_2\}$. This $d^2_{\mathcal{A}_k}$ matrix S, obtained as a — properly normalized and ordered — quantum "character table", defines the quantum analogue of a Fourier transform for the graphs $\mathcal{A}_k$. The matrix S is symmetric and such that $S^4 = 1$. In the opposite direction, the well known Verlinde formula [49] expresses fusion matrices N_λ in terms of coefficients of S:

$$\mathcal{N}_{\lambda\mu}^{\nu} = \sum_{\beta\in\mathcal{A}_k} \frac{S_{\lambda\beta}\, S_{\mu\beta}\, S^*_{\nu\beta}}{S_{0\beta}}, \tag{4}$$

where $\lambda = 0 = (0, 0)$ is the trivial representation. In the present paper we prefer to obtain the S matrix from the combinatorial data provided by the graph.

The modular generator T

The modular generator T is diagonal in the basis defined by vertices. Its eigenvalue associated with a vertex of shifted coordinates $\lambda = \{\lambda_1, \lambda_2\}$ is equal to [28]:

$$T_{\lambda\lambda} = \exp\left[2\mathrm{i}\pi\left(\frac{[\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2]-\kappa}{3\kappa}\right)\right]. \tag{5}$$

The square bracket in the numerator of the argument of exp can be simply read from the coordinates of the chosen vertex since it is the corresponding eigenvalue for the quadratic Casimir of the Lie group $SU(3)$. We describe as a "modular exponent" the whole numerator (i.e., the difference between the Casimir and the generalized Coxeter value κ) taken modulo 3κ. The T operator is therefore essentially (up to a trivial geometric phase) obtained as the exponential of the quadratic Casimir: The values for the shift $(-\kappa)$ and multiplicative constant (3κ) can indeed be fixed by imposing that the $SL(2, \mathbb{Z})$ relation $(ST)^3 = S^2$ hold.

The $SL(2, \mathbb{Z})$ representation defined by $\mathcal{A}_k$

Matrices S and T provide therefore a representation of the group $SL(2, \mathbb{Z})$ for each alcove of $SU(3)$. Actually, one obtains moreover the identity $T^{3\kappa} = 1$ so that this representation factorizes through the finite group $SL(2, \mathbb{Z}/3\kappa\mathbb{Z})$.

2.5. Symmetry and automorphism

The Z_3 action

Rotations of angle 0, $2\pi/3$ or $4\pi/3$ around the center of the equilateral triangle associated with the graph $\mathcal{A}_k$ define a Z_3 action — that we denote by z — on the set of vertices and therefore an endomorphism of the algebra (its cube is the identity). Its action on the irreps labelled by Dynkin labels (λ_1, λ_2) is given by:

$$z(\lambda_1, \lambda_2) = (k - \lambda_1 - \lambda_2, \lambda_1). \tag{6}$$

The Gannon automorphism ρ

It is defined on the vertices, as [22]

$$\rho = z^{kt}, \tag{7}$$

where t is the triality and k is the level of the graph. We found the following result [25]: If vertices v_1 and v_2 are such that $v_2 = \rho[v_1]$, then $T[v_1] = T[v_2]$. The proof is given in [25].

3. General properties of the $SU(3)$ system of graphs

This is a collection of graphs. As it will be discussed later, each graph G gives rise to a weak Hopf algebra (a quantum groupoïd) $\mathcal{B}G$, and each graph G is also associated with a given $\widehat{su}(3)$ modular invariant Z. At the moment, we suppose that the collection of graphs (also called the "Coxeter–Dynkin system of type $SU(3)$") is given and we list several of their properties. Several graphs (the orbifolds of the A series) were obtained by Kostov [31] but the full list of graphs for this system was obtained by Di Francesco and Zuber [17,16]. Later, A. Ocneanu, at the Bariloche school 2000 [40], explained why one member of their original list had to be removed.

3.1. First properties

Vertices and edges

Vertices of G are denoted as $a, b, c, \ldots$. Edges are oriented. In some cases there are multiple edges between two vertices.

Spectral properties of the graph G

A graph G belonging to the $SU(3)$ system is characterized by an adjacency matrix. Its biggest eigenvalue is called $\beta = 1 + 2\cos(2\pi/\kappa)$. The Coxeter number κ is read from β. The level is defined as $k = \kappa - 3$. The set of eigenvalues of the graph G is a subset of the eigenvalues of the graph $\mathcal{A}_k$ with same level. They are of the form $\beta(r_1, r_2)$ in Eq. (1), with possible multiplicities. The pairs of integers (r_1, r_2) are called the exponents of the graph G.

The associated modular invariant

$SU(3)$ graphs have been proposed as graphs associated with $\widehat{su}(3)$ modular invariant partition functions. These partition functions Z are sesquilinear forms on the characters labelled by irreps of $\widehat{su}(3)_k$. The correspondence is such that diagonal terms of Z match the set of exponents for the corresponding graph G. The interpretation for the off diagonal terms of Z was found by A. Ocneanu [39,38]. We shall come back to this later.

Quantum dimensions and order of G

One of the vertices of the graph G, denoted as $\mathbf{0}$, is called the unit vertex. It is defined from the eigenvector corresponding to β as the vertex associated with the smallest component.[4] The components of the normalized eigenvector associated with β (the dimension vector) define the quantum dimensions of the corresponding vertices — normalization is obtained by setting to 1 the quantum dimension of the unit vertex.[5] When there is only one arrow leaving (and going to) the unit vertex[6] $\mathbf{0}$, the quantum dimensions of its two neighbours (denoted as $\mathbf{1}$ and $\mathbf{1}^*$) are both equal to β. The sum of the squares of the quantum dimensions of vertices is called the order or the quantum mass of G, and denoted as $m(G)$.

3.2. The two representation theories associated with the bialgebra $\mathcal{B}G$

A quantum groupoïd $\mathcal{B}G$ is associated with any graph G of the $SU(3)$ system. It is both semi-simple and co-semi-simple. We present several basic properties here; more details will be given in Section 4.

The fusion algebra $A(G)$

The algebra $\mathcal{B}G$ endowed with its associative product is a direct sum of matrix algebras labelled by the index λ (i.e., by vertices of the $\mathcal{A}_k$ graph with same level). Its representation theory (algebra of characters) $A(G)$ is isomorphic to the fusion algebra of $\mathcal{A}_k$. Matrix representatives of the generators λ of $\mathcal{A}_k$ have already been introduced: They correspond to the fusion matrices N_λ.

The algebra of quantum symmetries $Oc(G)$

The dual algebra $\widehat{\mathcal{B}G}$ endowed with its associative product is also a direct sum of matrix algebras labelled by an index x. Its representation theory (algebra of characters) is called the "algebra of quantum symmetries" of

[4] If the graph possesses some (classical) symmetry, there can be several vertices associated with the smallest component. In those cases, we just choose one of them.

[5] It plays indeed the role of a unit when the graph G has self-fusion (see later), otherwise it is only a vertex whose quantum dimension is 1.

[6] This is for instance not so for $\mathcal{D}_k^*$.

G and denoted as $Oc(G)$. We call d_O the dimension of $Oc(G)$. It is an algebra with a unit (denoted as 0) and, for $SU(3)$ graphs, with – in general but not always – two algebraic generators (called chiral left and chiral right generators and denoted as 1_L and 1_R), together with their conjugates 1_L^* and 1_R^*. The Cayley graph of multiplication by the two generators 1_L and 1_R (two types of lines) is called the Ocneanu graph of G. The graph corresponding to the conjugated generators 1_L^* and 1_R^* is obtained from the (oriented) Ocneanu graph by reversing the arrows. $Oc(G)$ has also another conjugation, called the chiral conjugation, that permutes the two algebraic generators 1_L and 1_R. Another way of displaying the Cayley graph is to draw only the graph of multiplication by one chiral generator, say 1_L, and to associate (for example using dashed lines) each basis element with its chiral conjugate. Multiplication of a vertex x by the chiral generator 1_R is obtained as follows: We start with x, follow the dashed lines to find its chiral vertex y, then use the multiplication by 1_L and finally pull back using the dashed lines to obtain the result. Linear generators of $Oc(G)$ (i.e., vertices of the Ocneanu graph) that are identical with their chiral conjugates are called self-dual. The two subalgebras generated by the chiral generators are called chiral subalgebras. The intersection of these two subalgebras is called the ambichiral subalgebra, and its generators are the ambichiral generators (they are self-dual). $Oc(G)$, like $A(G) \simeq \mathcal{A}_k$, is not only an algebra but also an algebra that comes with a particular basis (the vertices of the Ocneanu graph), for which structure constants are non negative integers. The multiplication between vertices reads $x\,y = \sum_z O_{xy}^z\, z$, where O_{xy}^z, called quantum symmetry coefficients, are non-negative integers. Matrix representatives of these linear generators x of $Oc(G)$ are called "Ocneanu matrices" and denoted as O_x, with elements $(O_x)_{yz} = O_{xy}^z$. They form an anti-representation of the Ocneanu algebra:

$$O_x\, O_y = \sum_z O_{yx}^z\, O_z. \tag{8}$$

If $Oc(G)$ is commutative – which is not always the case – then $O_{xy}^z = O_{yx}^z$ and the Ocneanu matrices form a representation of the Ocneanu algebra: $O_x\, O_y = \sum_z O_{xy}^z O_z$. The structure of $Oc(G)$ is very much case dependent. One of the purposes of this paper is actually to present the corresponding results (for the $SU(3)$ system) in a synthetic way. In many cases $Oc(G)$ can be written as the direct sum of a chiral subalgebra and one or several modules over this subalgebra. Knowledge of the Ocneanu graph (i.e., the action of 1_L and 1_R) may sometimes be insufficient to encode the full structure (like for the D_4 case of the $SU(2)$ system). Matrices O_{1_L} and O_{1_R} are the adjacency matrices of the Ocneanu graph. The two dimension vectors (normalized eigenvectors associated with the largest eigenvalue for each adjacency matrix) allow one to attribute — unambiguously — quantum dimensions to all the linear generators of $Oc(G)$. Actually, the two chiral generators have dimension β and the whole list of quantum dimensions can be read directly from the Ocneanu graph by using the fact that this property is multiplicative, $q\,\dim(x\,y) = q\,\dim(x)\,q\,\dim(y)$. The sum of their squares is called the order or the quantum mass of $Oc(G)$, denoted as $m(Oc(G))$: It is equal to the order of $m(\mathcal{A}_k)$ of $\mathcal{A}_k = A(G)$. This property generalizes the usual group theory result.

3.3. *G as a module over* $A(G) = \mathcal{A}_k$

Call also G the vector space spanned by the vertices of a graph G. Call r the number of vertices of the graph. This vector space is a module for the action of the fusion algebra associated with $\mathcal{A}_k$, where k is the level of G (Coxeter number minus 3). The action is defined by the relation $\lambda\, a = \sum_b F_{\lambda a}^b\, b$, where $F_{\lambda a}^b$ are non-negative integers called fused or annular coefficients. In some cases, the same graph G may also be a module over some other graph of type A with a different Coxeter value, but we are not interested in this phenomenon.

Annular matrices

This action is encoded by a set of matrices F_λ called annular matrices or fused (not fusion!) matrices, defined by $(F_\lambda)_{ab} = F_{\lambda a}^b$. From the module property $\lambda\,(\mu\, a) = (\lambda\,\mu)\, a$, the annular matrices satisfy:

$$F_\lambda\, F_\mu = \sum_\nu N_{\lambda\mu}^\nu\, F_\nu. \tag{9}$$

They form a representation of the fusion algebra (usually of different dimension since $r \neq d_{\mathcal{A}_k}$). They are obtained by the same recurrence relation (2) as the fusion matrices but with $F_{(0,0)} = \mathbb{1}_{r\times r}$ and $F_{(1,0)} = Ad(G)$, where $Ad(G)$ is the adjacency matrix of G. We obtain in this way d_A matrices of size $r \times r$. As before d_A is the number of vertices of the associated $\mathcal{A}_k$ graph, the index λ of F_λ is a Young tableau.

Essential paths (also called horizontal paths)

Since annular matrices F_λ have non negative integer entries, one can associate a graph with every such matrix. If the matrix element of $(F_\lambda)_{ab} = p$, we introduce p oriented edges between vertices a and b of G. Such an edge is called an essential path of type λ from a to b. This graph will be called the horizontal graph of type λ. Remember that the λ index is a Young tableau (a vertex of the corresponding $\mathcal{A}_k$ diagram). The graph associated with the generator $F_{(1,0)}$ is the graph G itself.

Essential matrices (or horizontal matrices)

Essential matrices have the same information contents as the annular matrices, however, they are rectangular rather than square. They are defined as follows

$$(E_a)_{\lambda b} \doteq (F_\lambda)_{ab}. \tag{10}$$

We have therefore one essential matrix E_a for each vertex a of the graph G. The integer $(E_a)_{\lambda b}$ gives the number of horizontal paths of type λ from a to b. The property (9) can be written as follows using essential matrices:

$$N_\lambda\, E_a = E_a\, F_\lambda. \tag{11}$$

In particular we have $N_{(1,0)}\, E_0 = E_0\, F_{(1,0)}$. The essential matrix E_0 associated with the unit 0 of the graph G intertwines the adjacency matrices of the graphs G and $\mathcal{A}$: It is also called the $(\mathcal{A}_k, G)$ intertwiner.

Restriction–induction coefficients

Non-zero entries of the first line of F_λ (i.e. relative to the unit vertex of G) are called restriction coefficients. They define a restriction from $\mathcal{A}_k$ to G (like irreps of a group versus irreps of a subgroup). The branching rules are given by:

$$\lambda \hookrightarrow \sum_b (F_\lambda)_{1b}\, b = \sum_b (E_0)_{\lambda b}\, b. \tag{12}$$

The line indices corresponding to the non-zero entries of the column b of the matrix E_0 are called induction coefficients associated with the vertex b. They give the vertices λ for which b appears in their branching rules. The line indices (Young tableaux) corresponding to the non-zero entries of the first column of the matrix E_0 are called degrees of the family of would-be quantum invariant tensors by analogy with the situation that prevails for finite subgroups of Lie groups (for instance, when G is the fusion graph by the fundamental representation of binary polyhedral groups, these non-zero entries of the first column of E_0 reflect the existence of invariant symmetric tensors and therefore give the degrees of the Klein invariant polynomials for symmetry groups of Platonic bodies).

3.4. G as a module over $Oc(G)$

The vector space G is also a module for the action of the algebra of quantum symmetries $Oc(G)$. Call x the elements of $Oc(G)$. The action is defined by the relation $x\,a = \sum_b S_{xa}^b\, b$, where S_{xa}^b are non-negative integers called dual annular coefficients.

Dual annular matrices

The action can be encoded in a set of matrices S_x that we call the dual annular matrices, defined by $(S_x)_{ab} = S_{xa}^b$. From the module property $x\,(y\,a) = (x\,y)\,a$, the dual annular matrices satisfy:

$$S_x\, S_y = \sum_z O_{yx}^z\, S_z. \tag{13}$$

They satisfy the same relations as the Ocneanu matrices O_x (they form an anti-representation of the quantum symmetry algebra). We obtain in this way d_O matrices of size $r \times r$. As before d_O is the number of vertices of the associated Ocneanu graph.

Vertical paths

Since dual annular matrices S_x have non negative integer entries, one can associate a graph with every such matrix. If the matrix element $(S_x)_{ab} = p$, we introduce p oriented edges between vertices a and b of G. Such an edge is called a vertical path of type x from a to b. This graph will be called the vertical graph of type x. The vertical graphs associated with the two chiral generators of $Oc(G)$ coincide with G itself.

Vertical matrices

Vertical matrices have the same information content as the dual annular matrices, however, they are rectangular rather than square. They are defined as follows:

$$(R_a)_{xb} \doteq (S_x)_{ab}. \tag{14}$$

We have therefore one vertical matrix R_a for each vertex a of the graph G. The integer $(R_a)_{xb}$ gives the number of vertical paths of type x from a to b.

3.5. Self-fusion

$\mathcal{A}_k$ diagrams have self-fusion (the fusion algebra). A graph G has self-fusion when the vector space spanned by its vertices is not only a module over the corresponding $A(G)$ fusion algebra but when it possesses an associative algebra structure encoded by the graph itself (its adjacency matrix), with non negative integral structure constants, compatible with the already known $A(G)$ action. If $a, b, c, \ldots$ are vertices of a graph G with self-fusion, we have $a\, b = \sum_c G_{ab}^c\, c$, where the coefficients are non-negative integers. The unit $\mathbf{0}$ of the graph is the identity for the multiplication. The multiplication of some chosen vertex by the special vertex $\mathbf{1}$ (resp. $\mathbf{1}^*$) is given by the sum of vertices a such that there is an edge of G from the chosen vertex to a (resp. from a to the chosen vertex). The condition of compatibility between self-fusion and module structure reads $\lambda(a\, b) = (\lambda\, a)b$.

Conjugation

Conjugation is defined for all self-fusion graphs. It is compatible with the conjugation already defined for A graphs. We call a^* the conjugate of a in G. The compatibility condition is understood as follows: All vertices of $\mathcal{A}_k$ appearing in the induction list associated with a^* should be the conjugated vertices (taken in $\mathcal{A}_k$) of those associated with a. When these two sets are equal, then $a^* = a$. This provides a method for determining the conjugation of the G vertices. We have $(\lambda a)^* = \lambda^* a^*$, thus the annular coefficients should satisfy $(F_{\lambda^*})_{a^* b^*} = (F_\lambda)_{ab}$.

Triality

Triality is also defined for all graphs with self-fusion. It is compatible with the triality already defined for A graphs. This compatibility condition is understood as follows: If the level of the graph G is k, then all the vertices of $\mathcal{A}_k$ appearing in the induction list associated with a given vertex of G should have the same triality. This provides a method for determining the triality of the G vertices.

Graph matrices

The fusion of G vertices can be encoded in a set of matrices G_a with non-negative integer coefficients $(G_a)_{bc} = G_{ab}^c$, called graph matrices. We have $G_0 = F_{(0,0)}$, $G_1 = F_{(1,0)}$ and $G_{1^*} = F_{(0,1)}$. The compatibility condition for graphs with self-fusion (cf supra) reads $G_a\, F_\lambda = F_\lambda\, G_a$. In particular, using essential matrices E_a defined in Eq. (10) one can get $E_a = E_0\, G_a$.

Remark

Some of the graphs belonging to a Coxeter–Dynkin system have self-fusion, others do not. For example, in the $SU(2)$ system, the diagrams A_n, D_{even}, E_6 and E_8 have self-fusion, this is not the case for D_{odd} and E_7. In the $SU(3)$ system, diagrams $\mathcal{A}_k$, $\mathcal{D}_{3n}$, $\mathcal{E}_5$, $\mathcal{E}_9$ and $\mathcal{E}_{21}$ have self-fusion. The others do not.

Flatness

We believe that self-fusion is equivalent to flatness, as defined for instance in [35,36] or [29]. The two notions look a priori very different but it seems that all known graphs with self-fusion are also flat (and reciprocally). We are not aware of any formal proof relating the two concepts.

3.6. Coxeter–Dynkin systems of graphs, self-connections and Kuperberg spiders

A graph that is a member of a Coxeter–Dynkin system gives rise to a particular kind of quantum groupoïd. Such a graph is associated with some modular invariant, but sometimes more than one graph can be associated with the same

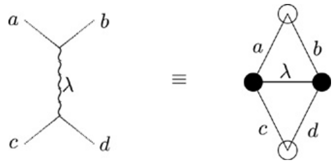

Fig. 1. A double triangle of type *GGAGG* of $\mathcal{B}$.

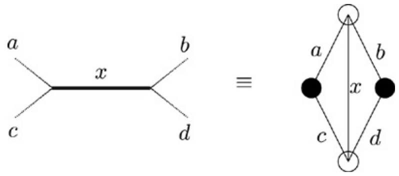

Fig. 2. A double triangle of type *GGOGG* of $\widehat{\mathcal{B}}$.

invariant. Moreover, a member of a Coxeter–Dynkin system has also to be compatible, in a sense that should be made precise, with a given Lie group (here $SU(3)$). Being a module over the graph algebra of a Weyl alcove at some level is a necessary but not sufficient condition. A condition, using the notion of self-connections on graphs, was given by A. Ocneanu in Bariloche (2000) [40] and this led him to discard one of the graphs of the original Di Francesco–Zuber list. We believe that the appropriate algebraic concept can be phrased in terms of Kuperberg spiders [30] but we have no rigorous proof that the two concepts are the same.

4. The quantum groupoïd associated with a pair (G_1, G_2)

If G_1 has self-fusion and if G_2 is a module over G_1, one can associate a bialgebra $\mathcal{B}(G_1, G_2)$ with this pair of graphs [39]. This bialgebra is a particular type of weak Hopf algebra (or quantum groupoïd) (see for instance [37,4, 5,34,33]). We call it the "Ocneanu quantum groupoïd" associated with the chosen pair. In particular if $G_2 = G$ and $G_1 = \mathcal{A}_k$, with k the level of G, we just define $\mathcal{B}G \doteq \mathcal{B}(\mathcal{A}_k, G)$, or simply $\mathcal{B}$ if the choice of G is clear from the context. In what follows we consider mostly bialgebras of that type.

4.1. The vector spaces $\mathcal{B}$ and $\widehat{\mathcal{B}}$

Admissible triangles

With every essential (i.e. horizontal) path of type λ between a and b one associates a triangle with one horizontal edge labelled by λ and two edges labelled by a and b. Such triangles (with 1 line of type A and 2 lines of type G) are called admissible triangles. By duality, they can also be drawn as (GGA) vertices. The vector space spanned by such triangles is called $Ess\mathrm{Path}(G)$ or $H\mathrm{paths}(G)$, it is graded by λ: $H\mathrm{paths}(G) = \sum_\lambda H\mathrm{paths}_\lambda(G)$.

With every vertical path of type x between a and b one associates a triangle with one vertical edge labelled by x and two edges labelled by a and b. Such triangles (with one line of type Oc and two lines of type G) are also called admissible triangles. By duality, they can also be drawn as (GGO) vertices. The vector space spanned by such triangles is called $V\mathrm{paths}(G)$, it is graded by x: $V\mathrm{paths}(G) = \sum_x V\mathrm{paths}_x(G)$.

Double triangles

We call $\mathcal{B}$ the graded vector space $\sum_\lambda H\mathrm{paths}_\lambda(G) \otimes H\mathrm{paths}_\lambda(G)$. It is spanned by double triangles $GGAGG$ (two triangles of type (GGA) sharing a common edge of type A). By duality they can also be drawn as diffusion diagrams (like in Fig. 1).

We call $\widehat{\mathcal{B}}$ the graded vector space $\sum_x V\mathrm{paths}_x(G) \otimes V\mathrm{paths}_x(G)$. It is spanned by double triangles $GGOGG$ (two triangles of type (GGO) sharing a common edge of type O). By duality they can also be drawn as diffusion diagrams (like in Fig. 2).

4.2. The multiplications

The multiplication $\circ$ on the vector space $\mathcal{B}$

This algebra structure on $\mathcal{B}$ is obtained by choosing the set of double triangles of type $(GGAGG)$ as a basis of matrix units e_{IJ} for an associative product that we call $\circ$, and such that multi-indices are like $\{I, J\} = \{(\lambda, a, b), (\lambda, c, d)\}$, i.e. with same λ.

The multiplication $\hat{\circ}$ on the dual vector space $\hat{\mathcal{B}}$

This algebra structure on $\hat{\mathcal{B}}$ is obtained by choosing the set of double triangles of type $(GGOGG)$ as a basis of matrix units ϵ^{AB} for an associative product that we call $\hat{\circ}$, and such that multi-indices are like $\{A, B\} = \{(x, a, b), (x, c, d)\}$, i.e. with same x.

Comultiplications and compatibility: Ocneanu cells

Since we have a product $\circ$ in $\mathcal{B}$ we have a coproduct $\hat{\Delta}$ in $\widehat{\mathcal{B}}$. Since we have a product $\hat{\circ}$ in $\widehat{\mathcal{B}}$ we have a coproduct Δ in $\mathcal{B}$. In order to have a bialgebra structure, we need a compatibility condition for the coproducts (homomorphism property). In order to ensure this, it is not possible to assume that the two bases of double triangles that we have used in $\mathcal{B}$ and in $\widehat{\mathcal{B}}$ are dual bases. On the contrary, the fact that there exists a non-trivial pairing (between these two bases) such that the compatibility conditions holds is the main non-trivial part of the claim that $\mathcal{B}$ is actually a bialgebra. This non-trivial pairing is determined by the family of Ocneanu cells or inverse cells $\langle \epsilon^{AB}, e_{IJ} \rangle$, labelled with tetrahedra a, b, λ, d, c, x (in some cases there is more than one path – horizontal or vertical – with fixed λ or x and given endpoints, so that cells may depend of other indices). Explicit determination of these numerical coefficients is not studied in the present paper.

For an arbitrary graph G, there are actually several (five) sets of such coefficients generalizing the Racah–Wigner $6j$ symbols; they obey orthogonality relations and several types (five) of mixed pentagonal relations. Their proper definition involves non-trivial normalization choices.

Scalar product and convolution product

Making a particular choice for a scalar product in $\mathcal{B}$, it is possible to trade the associative product $\hat{\circ}$, defined on the dual vector space $\widehat{\mathcal{B}}$ against an associative product $*$ (convolution product) in the vector space $\mathcal{B}$. The situation is self-dual so that we can also find a scalar product in $\widehat{\mathcal{B}}$ in order to trade the associative product $\circ$ defined on $\mathcal{B}$ against an associative product $\hat{*}$ in the dual vector space $\widehat{\mathcal{B}}$.

4.3. Properties of B

It is a finite dimensional semi-simple algebra and co-semi-simple coalgebra (equivalently, its dual $\widehat{\mathcal{B}}$ is also a finite dimensional semi-simple algebra and co-semi-simple coalgebra).

Quadratic sum rules

We call $d_\lambda = \dim(H\mathrm{Path}_\lambda)$ the dimensions of the blocks labelled by λ, associated with the first algebra structure, and $d_x = \dim(V\mathrm{Path}_x)$ the dimensions of those labelled by x, associated with the other algebra structure. Since the underlying vector space is the same, and since both algebra structures are semi-simple, we can calculate the dimension $d_{\mathcal{B}}$ of $\mathcal{B}$ in two possible ways and check the identity:

$$d_{\mathcal{B}} = \sum_\lambda d_\lambda^2 = \sum_x d_x^2. \tag{15}$$

The dimensions d_λ and d_x can be calculated from the annular and dual annular matrices: $d_\lambda = \sum_{a,b} (F_\lambda)_{ab}$, $d_x = \sum_{a,b} (S_x)_{ab}$.

Linear sum rules

Define $d_H = \sum_\lambda d_\lambda$ and $d_V = \sum_x d_x$. It happens that, in many cases, the relation $d_H = d_V$ holds, and when it does not, one knows how to correct it. The existence of this linear sum rule (first observed in [45]) is an observational fact. Its origin is not understood.

$\mathcal{B}$ is not a Hopf algebra but a weak Hopf algebra (a quantum groupoïd)

The main difference from the quantum group case is that the coproduct of the unit is not equal to the tensor square of the unit. What replaces it can be written as $\sum \mathbb{1}_{(1)} \otimes \mathbb{1}_{(2)}$. The terms appearing in this sum also show up in the axioms defining weak Hopf algebras (see for instance [4]). In particular the appropriate tensor product for the category of representations is not $\otimes$ but $\otimes \circ \Delta\mathbb{1}$.

Available references

The fact that a quantum groupoïd is associated with every member of a Coxeter–Dynkin system is not phrased as such in [39] but the two multiplicative structures are described there in quite general terms.[7] The correspondence between ADE graphs and particular weak Hopf algebras is also strongly suggested in [45]. Nowadays the fact that any member of a Coxeter–Dynkin system is associated with a quantum groupoïd (as defined by [4]) belongs to the folklore (see [42,6] for the description of this situation in the language of fusion categories and module categories). They are actually quantum groupoïds of a very particular kind (so they should better be called "Ocneanu quantum groupoïds"). In the case of the $SU(2)$ system, elementary proofs, based on axiomatic properties of Ocneanu cells, are now available in published form [14]; several explicit examples have also been worked out (for instance in [10] or [47]). In the case of the $SU(3)$ system, general proofs are not available. Our attitude in this paper is however to take the above property for granted.

5. The double fusion algebra and the modular splitting

5.1. Bimodule properties

Toric matrices and double annular matrices

The Ocneanu quantum groupoïds $\mathcal{B}G$ are of a very special kind. In particular, we have the following property involving simultaneously the two representation theories associated with the bialgebra $\mathcal{B}G$ — the fusion algebra $A(G)$ and the quantum symmetries algebra $Oc(G)$: $Oc(G)$ is an $A(G)$ bimodule, i.e., an $A(G)$–$A(G)$ module. This comes from the fact that in all cases, $Oc(G)$ can be written as the tensor square (maybe twisted or quotiented) of some graph algebra on which $A(G)$ acts. We write this action as $\lambda\, x\, \mu = \sum_y (V_{\lambda\mu})_{xy}\, y$. The $V_{\lambda\mu}$ are $d_O \times d_O$ matrices with non-negative integer coefficients, called double annular matrices. The same information can be encoded in $d_{\mathcal{A}_k} \times d_{\mathcal{A}_k}$ matrices W_{xy} called toric matrices, with non-negative integer coefficients defined by $(W_{xy})_{\lambda\mu} \doteq (V_{\lambda\mu})_{xy}$.

Double fusion equation

The bimodule associativity property $(\lambda\lambda')x(\mu\mu') = \lambda(\lambda' x \mu)\mu'$ leads to the following equation, called the double fusion equation:

$$V_{\lambda\mu}\, V_{\lambda'\mu'} = \sum_{\lambda''\mu''} N_{\lambda\lambda'}^{\lambda''}\, N_{\mu\mu'}^{\mu''}\, V_{\lambda''\mu''}. \tag{16}$$

This equation taken at $\mu = \mu' = 0$, at $\lambda = \lambda' = 0$ and at $\lambda' = \mu = 0$ leads to:

$$V_{\lambda 0}\, V_{\lambda' 0} = \sum_{\lambda''} N_{\lambda\lambda'}^{\lambda''}\, V_{\lambda'' 0} \tag{17}$$

$$V_{0\mu}\, V_{0\mu'} = \sum_{\mu''} N_{\mu\mu'}^{\mu''}\, V_{0\mu''} \tag{18}$$

$$V_{\lambda\mu'} = V_{\lambda 0}\, V_{0\mu'} = V_{0\mu'}\, V_{\lambda 0}. \tag{19}$$

Each set of matrices $V_{\lambda 0}$ or $V_{0\mu}$ gives therefore a representation of dimension $d_O \times d_O$ of the fusion algebra and V_{00} is the identity matrix. They can be determined from the same recurrence relation as the fusion matrices N_λ, once the fundamental generators $V_{(1,0),(0,0)}$ and $V_{(0,0),(1,0)}$ are known.

[7] This description is clearly related to the concept of (Ocneanu) paragroups introduced a long time before the notion of quantum groupoïd.

Other properties of $V_{\lambda\mu}$ matrices

The action is central. Writing $\lambda(x\ y)\mu = x(\lambda y\mu) = (\lambda x\mu)y$ leads to:

$$O_x\, V_{\lambda\mu} = V_{\lambda\mu}\, O_x = \sum_z (V_{\lambda\mu})_{xz}\, O_z. \tag{20}$$

The Ocneanu graph

With the set of relations satisfied by $V_{\lambda\mu}$ matrices and with the help of the known recurrence relations of irreps of $SU(3)$, all the coefficients $(V_{\lambda\mu})_{xy}$ can be simply determined from the fundamental matrices $V_{(1,0),(0,0)}$ and $V_{(0,0),(1,0)}$. These matrices are the adjacency matrices of the Ocneanu graph:

$$V_{(1,0),(0,0)} = O_{1_L} \qquad V_{(0,0),(1,0)} = O_{1_R}. \tag{21}$$

The Ocneanu graph determines (and is determined by) these two matrices.

Generalized partition functions

In the boundary conformal field theory associated with the given graph, the partition function on a torus with defect lines labelled by x and y is given by $Z_{xy} = \overline{\chi}\, W_{xy}\, \chi$ where χ is the vector of characters of affine $su(3)$ [44].

The modular matrix M

In particular, when there are no defect lines ($x = y = 0$), we recover the modular invariant partition function[8] $Z = \overline{\chi}\, M\, \chi$, since the modular invariant matrix $M = W_{00}$ commutes with the modular generators S and T in the representation of $SL(2, \mathbb{Z})$ associated with the Weyl alcove at this level. In contrast, the V_{00} matrix is the identity matrix.

The double intertwining relation

From the fact that a graph G with level k is an $\mathcal{A}_k$ module we deduced the intertwining relation given in Eq. (11), written in terms of essential matrices E_a attached to each vertex of the graph G. By analogy, let us introduce here the "essential tensor" K_x, with components $(K_x)_{\lambda\mu y} = (V_{\lambda\mu})_{xy}$, associated with each vertex x of $Oc(G)$. It can be written as a rectangular matrix of size $d_A^2 \times d_O$ (call it the double essential matrix). From the fact that $Oc(G)$ is an $A(G)$ bimodule, the double fusion Eq. (16) can be written using K_x, leading to the following double intertwining relation:

$$\tau \circ (N_\lambda \otimes N_\mu) K_x = K_x\, V_{\lambda\mu}, \tag{22}$$

where τ gives a flip on tensor components: $\tau \circ (T_{(\lambda'\lambda'')\,(\mu'\mu'')}) = T_{(\lambda'\mu')\,(\lambda''\mu'')}$.

Other useful formulae

We already recalled the graph interpretation for the diagonal entries of M in terms of exponents of the graph. More generally we have the following result [39,38]. The number of vertices d_O of the Ocneanu graph (also called "number of irreducible quantum symmetries") is equal to the sum of squares of entries of the modular matrix. Moreover, the algebra of quantum symmetries is isomorphic to a direct sum of finite dimensional matrix algebras of the form $\bigoplus_{m,n} \mathrm{Mat}_{M_{mn}}(C)$ where M_{mn} are the entries of the modular matrix. In other words these entries give the dimensions of the irreducible representations of this algebra.

Another interpretation for these numerical entries can be given in terms of higher quantum Klein invariants (cf supra).

The above result was stated, by A. Ocneanu, for the $SU(2)$ system. It can also be checked explicitly for all members of the $SU(3)$ system. In the framework of the theory of sectors, such a decomposition has been proved in Theorem 6.8 of [3] in a completely general setting (theorem 5.3 of the same paper shows that it is equivalent to Ocneanu's graphical method). A nice graphical way to encode the modular matrix M associated with a graph G is provided by the "modular diagram": It is a picture of the Weyl chamber at the given level, with arcs connecting the vertices associated with non-zero entries M_{mn}. The degrees of quantum invariant tensors can also be read from this diagram:

[8] We write in this paper $\mathcal{Z} = \sum_{\lambda,\mu} \chi_\lambda\, M_{\lambda\mu}\, \overline{\chi}_\mu$. If we had written $M_{\lambda\overline{\mu}}$ instead of $M_{\lambda\mu}$, the definitions of what we call G and G^* graphs should be flipped.

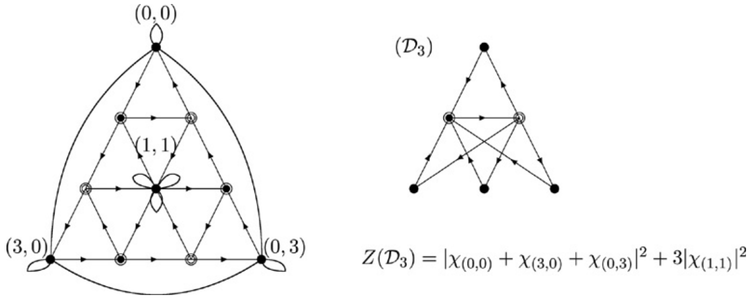

Fig. 3. The modular diagram and the modular invariant associated with the $\mathcal{D}_3$ graph.

They correspond to those vertices that belong to the arc going though the origin (0,0). For instance Fig. 3 shows these results for the $\mathcal{D}_3$ case.

The first part of the previous theorem can be written $d_O = \mathrm{Tr}(M M^\dagger)$. When the modular splitting technique (see the next section) is used to determine explicitly the W_{xy} and the algebra $Oc(G)$ itself, the above result[9] provides a numerical check.

5.2. Modular splitting

The double fusion Eq. (16) at $x = y = 0$ leads to the following equation, written in terms of W matrices, called the modular splitting equation:

$$\sum_z (W_{0z})_{\lambda\mu} (W_{z0})_{\lambda'\mu'} = \sum_{\lambda''\mu''} (N_\lambda)_{\lambda'\lambda''} (N_\mu)_{\mu'\mu''} M_{\lambda''\mu''}. \tag{23}$$

The double fusion Eq. (16) at $y = 0$ leads to the following equation, written in terms of W matrices, called the generalized modular splitting equation:

$$\sum_z (W_{xz})_{\lambda\mu} (W_{z0})_{\lambda'\mu'} = \sum_{\lambda''\mu''} (N_\lambda)_{\lambda'\lambda''} (N_\mu)_{\mu'\mu''} (W_{x0})_{\lambda''\mu''}. \tag{24}$$

Modular splitting technique I: From the modular matrix M to the toric matrices W_{x0}

In many cases, the graph G itself is not known (see comments in the last section) and the only knowledge that we have is the modular matrix M. It is possible to use the modular splitting equation to determine the toric matrices. This was certainly the road followed by A. Ocneanu but a general method of resolution was first described in [11], many more details and examples can be found in [27].

One starts from the modular splitting Eq. (23). The fusion matrices N_λ and the modular matrix M are known. The right hand side of (23) is thus known: It can be seen as a matrix, called K (the "fused modular matrix"), of size $d_A^2 \times d_A^2$. Toric matrices that appear on the left hand side are integer entry matrices $d_A \times d_A$ to be determined. The number of distinct toric matrices with one twist is equal to the rank of K. In simple cases, the number $d_O = Tr(MM^\dagger)$ of Ocneanu generators O_x is precisely equal to the rank of K. In more complicated cases the rank of K is strictly smaller (which means that several toric matrices associated with distinct generators O_x may coincide). The explicit method leading to the determination of toric matrices (i.e., the technique used to solve the modular splitting equation) is not explained in the present paper. It is described (for a particular example) in one section of [11]. A detailed study of this method together with several $SU(3)$ examples will be given in [27].

Modular splitting technique II: From the toric matrices W_{x0} to the Ocneanu generators O_x

Once we have determined the toric matrices with one twist W_{x0}, we have to determine the toric matrices W_{xy}. The right hand side of the generalized modular splitting Eq. (24) is known. Toric matrices W_{xy} appearing on the left hand side can then be calculated. This is equivalent to solve the double intertwining relation (22) in the particular case $x = 0$ (this is a set of linear equations that involves only the already determined toric matrices with only one twist).

[9] Here – and in the whole paper – we have in mind the simply laced cases (the *ADE* diagrams) or their generalizations.

This leads therefore to the determination of the double annular matrices and in particular of the two chiral generators O_{1_L} and O_{1_R}. The other Ocneanu generators O_x can be determined solving Eq. (20).

Remark

Once the algebra (or graph) of quantum symmetries $Oc(G)$ has been obtained, we can determine the generalized Dynkin diagram G as a module graph on $Oc(G)$. Sometimes there is not unicity of the result and two different graphs may be associated with the same initial modular invariant. See also our comments in the last section.

Relative modular splitting formula and relative double fusion algebra

Often, the algebra $Oc(G)$ is not only a bimodule over $A(G)$ but also a bimodule over the graph algebra of H where H is a graph with self-fusion on which $A(G)$ acts. In the cases where G admits self-fusion, it is often the case that H is G itself. In those cases we have a relative modular splitting formula: Fusion matrices are still the same but the relative modular matrix M^{rel} is written in terms of the G graph (so it is of size d_G^2 rather than d_A^2); $M = E_0\, M^{\mathrm{rel}}\, E_0^T$, where E_0 is the first essential matrix (intertwiner). In the same way, toric matrices W of size d_A^2 are replaced by relative toric matrices W^{rel} of size d_G^2. The modular splitting technique can be applied as before, with the advantage that the size of tensors is greatly reduced. Once the relative matrices are found, we can retrieve the others using the relation $W_{xy} = E_0 {W^{\mathrm{rel}}}_{xy} E_0^T$. Such an example is worked out in the last section of Ref. [11]

5.3. A dual bimodule structure?

Axioms for quantum groupoïds are certainly self-dual, but the objects that we have at hand are not generic: They are quite special. In particular, if it is clear that $Oc(G)$ is an $A(G)$ bimodule, there is no obvious reason for $A(G)$ to be an $Oc(G)$ bimodule. If it were so, this action would be defined by a set of coefficients P_{xy}, with $x\,\lambda\,y = \sum_\mu (P_{xy})_{\lambda\mu}\,\mu$. The P_{xy} being of dimension $d_{\mathcal{A}_k} \times d_{\mathcal{A}_k}$ and the bimodule associativity property $(xx')\lambda(yy') = x(x'\lambda y)y'$ would lead to a double quantum symmetry equation: $P_{x'y}\,P_{xy'} = \sum_{x''y''} O_{xx'}^{x''}\,O_{yy'}^{y''}\,P_{x''y''}$. This equation taken at $y = y' = 0$, at $x = x' = 0$ and at $x = y = 0$ would itself lead to: $P_{x'0}\,P_{x0} = \sum_{x''} O_{xx'}^{x''}\,P_{x''0}$, $P_{0y}\,P_{0y'} = \sum_{y''} O_{yy'}^{y''}\,P_{0y''}$, $P_{x'y'} = P_{x'0}\,P_{0y'} = P_{0y'}\,P_{x'0}$ and each set of matrices P_{x0} or P_{0y} would give respectively an anti-representation and a representation of dimension $d_{\mathcal{A}_k} \times d_{\mathcal{A}_k}$ of the quantum symmetry algebra. Now, what could these P_{xy} matrices be? One obvious candidate is to set them equal to the toric matrices W_{xy}. The problem is that this choice cannot work since, as can be checked on simple examples, $W_{x'y'}$ is not equal to $W_{x'0}W_{0y'}$ in general. Existence of a dual bimodule structure is not excluded, but if it exists, it cannot be defined by the toric matrices alone. Supposing the existence of such dual bimodule structure, it should also satisfy some compatibility conditions, like $(\lambda(x(\mu(y\,a)))) = ((\lambda\,x\,\mu)(y\,a)) = (\lambda(x\,\mu\,y)a))$, leading to the following set of relations:

$$S_y\,F_\mu\,S_x\,F_\lambda = \sum_z (V_{\lambda\mu})_{xz}\,S_y\,S_z = \sum_\nu (P_{xy})_{\mu\nu}\,F_\nu\,F_\lambda. \tag{25}$$

5.4. Realization of the Ocneanu quantum symmetries

In many cases $Oc(G)$ can be written in terms of the tensor square of the graph algebras of some related graph K with self-fusion, with the tensor product taken over a subalgebra, called the modular subalgebra J. In the simplest cases, i.e., when G has self-fusion, K is G itself. The set of elements of J is determined by modular properties [9,12,13,47]. Each vertex of an $\mathcal{A}_k$ graph has a fixed modular operator value T. The vector space spanned by vertices of a G graph is a module over $\mathcal{A}_k$, and one can try to define a modular operator value on vertices of G. Suppose that the vertex a of G appears in the branching rules (restriction map from $\mathcal{A}_k$ to G) of both vertices λ and μ of $\mathcal{A}_k$. The vertex a will have a well-defined modular operator value if the two values $T(\lambda)$ and $T(\mu)$ are equal. The set of vertices having this property is a subalgebra of the graph algebra of G, denoted as J.

As already commented, non-trivial multiplicities in the modular matrix lead to non-commutativity for $Oc(G)$. This happens whenever G possesses classical symmetries.[10] In those cases, the algebraic realization of $Oc(G)$ involves not

[10] By this we mean that, the unit vertex being chosen, the graph still contains a classical symmetry, making impossible a direct computation of the table of multiplication.

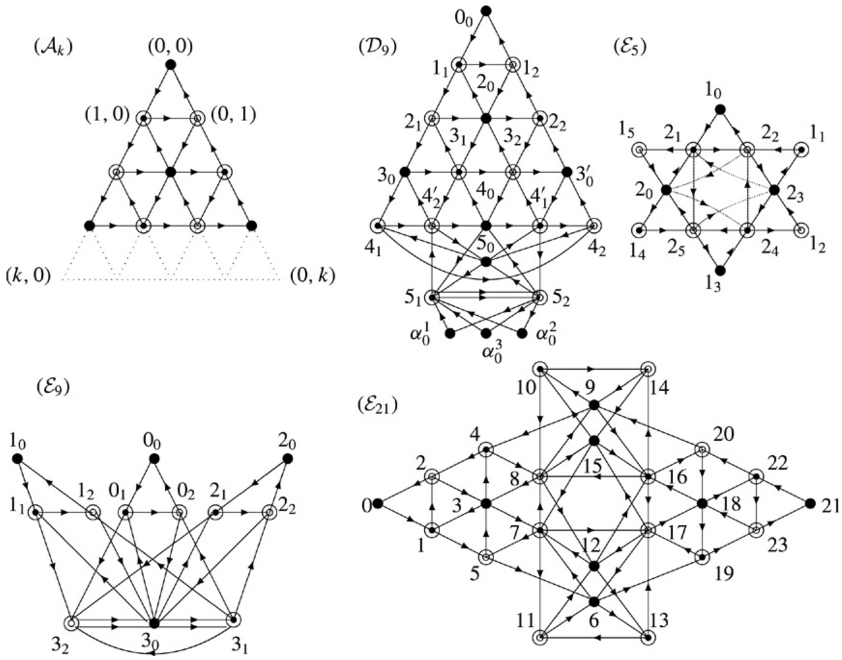

Fig. 4. Some graphs with self-fusion: The $\mathcal{A}_k$ series, $\mathcal{D}_9$, $\mathcal{E}_5$, $\mathcal{E}_9$ and $\mathcal{E}_{21}$.

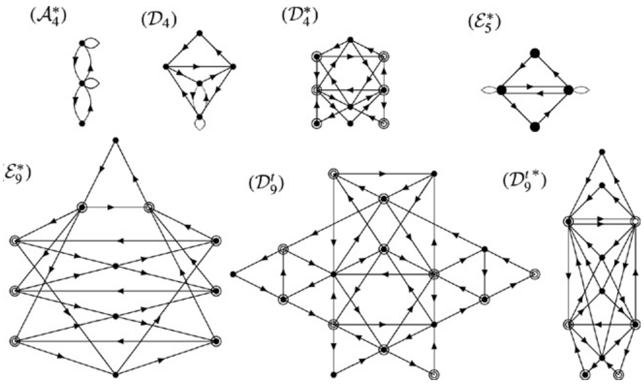

Fig. 5. Some module graphs without self-fusion: $\mathcal{A}_4^*$, $\mathcal{D}_4$, $\mathcal{D}_4^*$, $\mathcal{E}_5^*$, $\mathcal{E}_9^*$, $\mathcal{D}_9^t$, and $\mathcal{D}_9^{t^*}$.

only a tensor square over some subalgebra but also a cross-product by an appropriate discrete group algebra [47]. The bimodule structure of $Oc(G)$ over $\mathcal{A}_k \otimes \mathcal{A}_k$ is thus related to the module structure of G over $\mathcal{A}_k$.

6. The $SU(3)$ system of graphs and their quantum symmetries

Starting with the complete list of modular invariants [22], the list of graphs was found by [17], slightly amended by [40]. We believe that a determination of the graph of quantum symmetries associated with the above was worked out in 2000 or before by A. Ocneanu (unpublished). We now present a compendium of results concerning not only these quantum symmetries but also several other results that use the concepts introduced in previous sections. In particular we give in most cases an algebraic realization of $Oc(G)$ that allows one to perform calculations without having to use the graph of quantum symmetries. A detailed study of several cases has already been made available in the literature [13,47] and details concerning the others will be published elsewhere [25,27,24]. Some of these graphs are displayed in Figs. 4 and 5.

6.1. The $\mathcal{A}$ series and its conjugated series

6.1.1. The $\mathcal{A}$ series (graphs with self-fusion)

The $\mathcal{A}_k$ graphs are the Weyl alcoves of $SU(3)$ at level k. We have $A(\mathcal{A}_k) = \mathcal{A}_k$, so the annular matrices coincide with the fusion matrices: $F_\lambda = N_\lambda$. The algebra of quantum symmetries is realized as $Oc(\mathcal{A}_k) = \mathcal{A}_k \dot{\otimes} \mathcal{A}_k$ where the tensor product is taken over $\mathcal{A}_k$ with the identification $\lambda \dot{\otimes} \mu \equiv \lambda\mu^* \dot{\otimes} 0$. A basis of $Oc(\mathcal{A}_k)$ is $x = \lambda \dot{\otimes} 0$ and the dimension $d_O = d_{\mathcal{A}_k}$. The dual annular matrices are $S_x = F_\lambda = N_\lambda$ and the double annular matrices are $V_{\lambda\mu} = N_\lambda N_{\mu^*}$. The modular invariant associated with the $\mathcal{A}_k$ graph is diagonal $M_{\lambda\mu} = \delta_{\lambda\mu}$. We can easily check that $(V_{\lambda\mu})_{00} = M_{\lambda\mu}$. The two algebras $\mathcal{BA}_k$ and $\widehat{\mathcal{B}}\mathcal{A}_k$ are isomorphic. We have $d_x = d_\lambda$, the quadratic and linear sum rules are trivially satisfied. In the $SU(2)$ system, i.e. for ADE diagrams, the value of $d_H = \sum_\lambda d_\lambda$ has been obtained independently, for all diagrams, by A. Ocneanu (unpublished) and by [45] and [9,12]. It is easy to see, for instance, that for $A_{k+1} = \mathcal{A}_k$ graphs, the following formula holds : $d_H = \frac{(k+1)(k+2)(k+3)}{6}$. Actually, setting $r = k+1$ (the number of vertices) and $\kappa = k+2$ (the usual Coxeter number), this formula also works for D-graphs and for exceptionals, when it is written as $d_H = r\kappa(\kappa+1)/6$. It is interesting to notice that the same formula also gives the dimension of the Gelfand–Ponomarev preprojective algebra associated with the chosen graph (see [32]). For the $SU(3)$ system of graphs (now $\kappa = k+3$) we observe that the dimension $d_H = \sum_\lambda d_\lambda$ of graphs $\mathcal{A}_k$ is given by the formula

$$d_H = \frac{(k+1)(k+2)(k+3)(k+4)(k+5)(k^2+6k+14)}{1680}. \tag{26}$$

6.1.2. The $\mathcal{A}^*$ modules (no self-fusion)

The $\mathcal{A}_k^*$ graphs are the conjugated graphs of $\mathcal{A}_k$. Their vertices are the real vertices of $\mathcal{A}_k$ (see for example [21, 46,1]). We have $A(\mathcal{A}_k^*) = \mathcal{A}_k$. The algebra of quantum symmetries is realized as $Oc(\mathcal{A}_k^*) = \mathcal{A}_k \dot{\otimes} \mathcal{A}_k$ where the tensor product is again taken over $\mathcal{A}_k$ but now with the identification $\lambda \dot{\otimes} \mu \equiv \lambda\mu \dot{\otimes} 0$. A basis of $Oc(\mathcal{A}_k^*)$ is again $x = \lambda \dot{\otimes} 0$, and we have $d_O = d_{\mathcal{A}_k}$. The dual annular matrices are $S_x = F_\lambda$ and the double annular matrices are $V_{\lambda\mu} = N_\lambda N_\mu$. The modular invariant is $M_{\lambda\mu} = \delta_{\lambda\mu^*}$. The two algebras $\mathcal{BA}_k$ and $\widehat{\mathcal{B}}\mathcal{A}_k$ are isomorphic. We have $d_x = d_\lambda$, the quadratic and linear sum rules are trivially satisfied.

6.2. The $\mathcal{D}$ series and the conjugated $\mathcal{D}^*$ series

The $\mathcal{D}_k = \mathcal{A}_k/3$ graphs are orbifold graphs of the $\mathcal{A}_k$ graphs. They are obtained from the action of the geometrical $\mathbb{Z}_3$-automorphism z (see Eq. (6)) on irreps of the $\mathcal{A}_k$ graphs [31,19,17]. Vertices of $\mathcal{A}_k$ that belong to the same orbit lead to a single vertex in the orbifold graph $\mathcal{D}_k$. When there is a fixed vertex under z (this happens when $k = 0 \bmod 3$), this vertex is triplicated on the orbifold graph. Among all orbifold graphs $\mathcal{D}_k$, the $\mathcal{D}_{3n}$ are the only ones that have self-fusion.

6.2.1. The $\mathcal{D}_k$ orbifold modules for $k \neq 0$ mod 3 (no self-fusion)

For $k \neq 0$ mod 3, the $\mathcal{D}_k$ graphs have $(k+1)(k+2)/6$ vertices. One can define a graph algebra with non negative integer structure constants for these graphs, but it is not compatible with the $\mathcal{A}_k$ action. Therefore these graphs do not have self-fusion. The Ocneanu algebra is realized as $Oc(\mathcal{D}_k) = \mathcal{A}_k \dot{\otimes} \mathcal{A}_k$ where the tensor product is again taken over $\mathcal{A}_k$ but with the identification $\lambda \dot{\otimes} \mu \equiv \lambda\rho(\mu^*) \dot{\otimes} 0$, where ρ is the Gannon twist (see Eq. (7)). A basis of $Oc(\mathcal{D}_k)$ is $x = \lambda \dot{\otimes} 0$, and we have $d_O = d_{\mathcal{A}_k}$. The dual annular matrices are $S_x = F_\lambda$ and the double annular matrices are $V_{\lambda\mu} = N_\lambda N_{\rho(\mu^*)}$. The associated modular invariant is $M_{\lambda\mu} = \delta_{\lambda\rho(\mu)}$. The two algebras $\mathcal{BD}_k$ and $\widehat{\mathcal{B}}\mathcal{D}_k$ are isomorphic. We have $d_x = d_\lambda$, the quadratic and linear sum rules are trivially satisfied. The dimensions d_λ of the blocks labelled by λ (or by x, which is the same here) satisfy $d_\lambda(\mathcal{D}_k) = d_\lambda(\mathcal{A}_k)/3$. The dimensions therefore satisfy $\dim(\mathcal{BD}_k) = \dim(\mathcal{BA}_k)/9$.

6.2.2. The $\mathcal{D}_k^*$ conjugated orbifold modules $k \neq 0$ mod 3 (no self-fusion)

The conjugated orbifold graphs $\mathcal{D}_k^*$ are the unfolded (i.e. triplicated) graphs of the $\mathcal{A}_k^*$ ones [17], i.e. their adjacency matrices are such that $Ad\left(\mathcal{D}_k^*\right) = \sigma_{123} \otimes Ad\left(\mathcal{A}_k^*\right)$, where $\sigma_{123} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is the permutation matrix. These graphs are modules over the fusion algebras $\mathcal{A}_k$. The Ocneanu algebra is realized as $Oc(\mathcal{D}_k^*) = \mathcal{A}_k \dot{\otimes} \mathcal{A}_k$ where the tensor product is again taken over $\mathcal{A}_k$ but with the identification $\lambda \dot{\otimes} \mu \equiv \lambda\rho(\mu) \dot{\otimes} 0$, where ρ is the Gannon twist defined in Eq. (7). A basis of $Oc(\mathcal{D}_k^*)$ is again $x = \lambda \dot{\otimes} 0$, and we have $d_O = d_{\mathcal{A}_k}$. The dual annular matrices are $S_x = F_\lambda$ and the double annular matrices are $V_{\lambda\mu} = N_\lambda N_{\rho(\mu)}$. The associated modular invariant is $M_{\lambda\mu} = \delta_{\lambda\rho(\mu^*)}$. The two algebras $\mathcal{B}\mathcal{D}_k^*$ and $\widehat{\mathcal{B}}\mathcal{D}_k^*$ are isomorphic. We have $d_x = d_\lambda$, the quadratic and linear sum rules are trivially satisfied. The dimensions d_λ of the blocks labelled by λ (or by x, which is the same here) satisfy $d_\lambda(\mathcal{D}_k^*) = 3\, d_\lambda(\mathcal{A}_k^*)$. The dimensions therefore satisfy $\dim(\mathcal{B}\mathcal{D}_k^*) = 9 \dim(\mathcal{B}\mathcal{A}_k^*)$.

6.2.3. *The $\mathcal{D}_k$ orbifolds for $k = 0$ mod 3 (self-fusion)*

For $k = 0$ mod 3, the $\mathcal{A}_k$ graphs have a central vertex $\mathbb{Z}_3$-invariant, which is triplicated on the orbifold graph. In this case $\mathcal{D}_k$ graphs have $(\frac{(k+1)(k+2)}{2} - 1)/3 + 3$ vertices, and they possess self-fusion. The associated modular invariant partition function is:

$$\mathcal{Z}(\mathcal{D}_k) = \frac{1}{3} \sum_{\lambda | t(\lambda)=0} |\chi_\lambda^k + \chi_{z(\lambda)}^k + \chi_{z^2(\lambda)}^k|^2. \tag{27}$$

The special vertex z-invariant on the $\mathcal{A}_k$ graph leads to the presence of a coefficient equal to 3 in the modular invariant. Therefore the algebra of quantum symmetries of $\mathcal{D}_{3n}$ is non-commutative. A realization is given by a semi-direct product $Oc(\mathcal{D}_{3n}) = \mathcal{D}_{3n} \ltimes \mathbb{Z}_3$ (see [48]). The Ocneanu graph is made of three copies of the $\mathcal{D}_{3n}$ graph, its dimension is $d_O = (k+1)(k+2)/2 + 8$. The quadratic sum rule is satisfied but the linear sum rule does not hold $d_H \neq d_V$ (it may be recovered by introducing appropriate symmetry factors).

6.2.4. *The $\mathcal{D}_k^*$ conjugate orbifolds for $k = 0$ mod 3 (no self-fusion)*

The conjugate orbifold graphs $\mathcal{D}_k^*$ are the unfolded (i.e. triplicated) graphs of the $\mathcal{A}_k^*$ ones [17]. These graphs are modules over the fusion algebras $\mathcal{A}_k$. For $k = 0$ mod 3, the associated modular invariant partition function is

$$\mathcal{Z}(\mathcal{D}_k^*) = \frac{1}{3} \sum_{\lambda | t(\lambda)=0} (\chi_\lambda^k + \chi_{z(\lambda)}^k + \chi_{z^2(\lambda)}^k) \left(\overline{\chi_{\lambda^*}^k} + \overline{\chi_{z(\lambda)^*}^k} + \overline{\chi_{z^2(\lambda)^*}^k}\right). \tag{28}$$

Its algebra of quantum symmetries is also non-commutative, and can be realized as a conjugated version of semi-direct product $Oc(\mathcal{D}_{3n}) = \mathcal{D}_{3n} \ltimes \mathbb{Z}_3$ (see [48]). Its dimension is $d_O(\mathcal{D}_k^*) = d_O(\mathcal{D}_k)$. The quadratic sum rule is satisfied but the linear sum rule does not hold $d_H \neq d_V$ (it may be recovered by introducing appropriate symmetry factors).

6.3. *Exceptional graphs with self-fusion and their modules*

In the $SU(3)$ family, we have three exceptional graphs with self-fusion, namely $\mathcal{E}_5$, $\mathcal{E}_9$ and $\mathcal{E}_{21}$. Diagrams $\mathcal{E}_5$ and $\mathcal{E}_{21}$ are generalizations of the two Dynkin diagrams E_6 and E_8. We have also the module graphs $\mathcal{E}_5^* = \mathcal{E}_5/3$ and $\mathcal{E}_9^* = \mathcal{E}_9/3$ (they do not have self-fusion). Finally we have the exceptional graph $\mathcal{D}_9^t$ obtained from the exceptional twist of the $\mathcal{D}_9$ graph (a generalization of the E_7 Dynkin diagram), together with the conjugated exceptional graph $\mathcal{D}_9^{t^*}$.

6.3.1. *The exceptional $\mathcal{E}_5$ graph (self-fusion)*

The $\mathcal{E}_5$ graph has self-fusion and has 12 vertices denoted as 1_i and 2_j where $i, j = 1, 2, \ldots, 6$. The unit vertex is 1_0 and the fundamental conjugated generators are 2_1 and 2_2 (for more details see [13] and [47]). Its quantum mass is $m(\mathcal{E}_5) = 12(2 + \sqrt{2})$. The associated modular invariant partition functions is:

$$\begin{aligned}\mathcal{Z}(\mathcal{E}_5) = {} & |\chi_{(0,0)}^5 + \chi_{(2,2)}^5|^2 + |\chi_{(0,2)}^5 + \chi_{(3,2)}^5|^2 + |\chi_{(2,0)}^5 + \chi_{(2,3)}^5|^2 \\ & + |\chi_{(2,1)}^5 + \chi_{(0,5)}^5|^2 + |\chi_{(3,0)}^5 + \chi_{(0,3)}^5|^2 + |\chi_{(1,2)}^5 + \chi_{(5,0)}^5|^2.\end{aligned}$$

The modular subalgebra is $J = \{1_i, i = 1, \ldots, 6\}$ and a realization of the Ocneanu algebra is given by $Oc(\mathcal{E}_5) = \mathcal{E}_5 \dot{\otimes}_J \mathcal{E}_5$, with the identifications $a \dot{\otimes}_J u\, b \equiv a\, u^* \dot{\otimes}_J b$, for all $u \in J$ and $a, b \in \mathcal{E}_5$. Conjugation on $\mathcal{E}_5$ is defined as: $1_0{}^* = 1_0$, $1_5{}^* = 1_1$, $1_4{}^* = 1_2$, $1_3{}^* = 1_3$, $2_0{}^* = 2_3$, $2_1{}^* = 2_2$ and $2_5{}^* = 2_4$ (it corresponds to the symmetry with respect to the vertical axis joining vertices 1_0 and 1_3 of the diagram $\mathcal{E}_5$ given on Fig. 4). Its dimension is

24 and a basis of $Oc(\mathcal{E}_5)$ is given by $a\,\dot{\otimes}_J\,1_0$ and $b\,\dot{\otimes}_J\,2_0$, for $a, b \in \mathcal{E}_5$. The chiral generators are $2_1\,\dot{\otimes}_J\,1_0$ and $1_0\,\dot{\otimes}_J\,2_1 \equiv 1_5\,\dot{\otimes}_J\,2_0$. The left and right chiral subalgebras are $L = \{a\,\dot{\otimes}_J\,1_0\}$ and $R = \{1_0\,\dot{\otimes}_J\,a\}$. The quantum mass of $Oc(\mathcal{E}_5)$ is $m\,[Oc(\mathcal{E}_5)] = \frac{m[\mathcal{E}_5].m[\mathcal{E}_5]}{m[J]} = m\,[\mathcal{A}_5] = 48\left(3+\sqrt{2}\right)$. The linear and quadratic sum rules hold and read $d_H = d_V = 720$, $\dim(\mathcal{B}\mathcal{E}_5) = 29\,376$, respectively.

6.3.2. The exceptional module of the $\mathcal{E}_5$ graph (no self-fusion)

The $\mathcal{E}_5^* = \mathcal{E}_5/3$ is the $\mathbb{Z}_3$-orbifold graph of $\mathcal{E}_5$, it has four vertices. It is a module over $\mathcal{A}_5$ and over $\mathcal{E}_5$. In particular it has the same norm $\beta = [3]_q = 1+\sqrt{2}$ as $\mathcal{A}_5$ and $\mathcal{E}_5$. Its quantum mass is $m(\mathcal{E}_5^*) = m(\mathcal{E}_5)/3 = 4(2+\sqrt{2})$. The associated modular invariant partition function is:

$$\begin{aligned}\mathcal{Z}(\mathcal{E}_5^*) &= |\chi^5_{(0,0)} + \chi^5_{(2,2)}|^2 + |\chi^5_{(3,0)} + \chi^5_{(0,3)}|^2 + (\chi^5_{(0,2)} + \chi^5_{(3,2)})\left(\overline{\chi^5_{(2,0)}} + \overline{\chi^5_{(2,3)}}\right)\\ &\quad + (\chi^5_{(2,0)} + \chi^5_{(2,3)})\left(\overline{\chi^5_{(0,2)}} + \overline{\chi^5_{(3,2)}}\right) + (\chi^5_{(1,2)} + \chi^5_{(5,0)})\left(\overline{\chi^5_{(0,5)}} + \overline{\chi^5_{(2,1)}}\right)\\ &\quad + (\chi^5_{(2,1)} + \chi^5_{(0,5)})\left(\overline{\chi^5_{(1,2)}} + \overline{\chi^5_{(5,0)}}\right).\end{aligned}$$

The Ocneanu algebra is $Oc(\mathcal{E}_5^*) = \mathcal{E}_5\,\dot{\otimes}_J\,\mathcal{E}_5$ where the tensor product is taken over the modular subalgebra J of $\mathcal{E}_5$ but with the identifications $a\,\dot{\otimes}_J\,u\,b \equiv a\,u\,\dot{\otimes}_J\,b$, for all $u \in J$ and $a, b \in \mathcal{E}_5$. The two algebras $Oc(\mathcal{E}_5)$ and $Oc(\mathcal{E}_5^*)$ are isomorphic but their realization in terms of tensor products are different. Here the right chiral generator is $1_0\,\dot{\otimes}_J\,2_1 \equiv 1_1\,\dot{\otimes}\,2_0$. The quantum mass is $m(Oc(\mathcal{E}_5^*)) = m(Oc(\mathcal{E}_5))$. The dimensions of the blocks labelled by λ and x satisfy $d_\lambda(\mathcal{E}_5^*) = d_\lambda(\mathcal{E}_5)/3$ and $d_x(\mathcal{E}_5^*) = d_x(\mathcal{E}_5)/3$. The linear and quadratic sum rules hold and read $d_H = d_V = 720/3 = 240$ and $\dim(\mathcal{B}\mathcal{E}_5^*) = \dim(\mathcal{B}\mathcal{E}_5)/9 = 3\,264$.

6.3.3. The exceptional $\mathcal{E}_9$ graph (self-fusion)

The $\mathcal{E}_9$ graph has self-fusion and possesses 12 vertices denoted as 0_i, 1_i, 2_i and 3_i where $i = 0$, 1 or 2. Its quantum mass is $m(\mathcal{E}_9) = 36(2+\sqrt{3})$. The associated modular invariant partition function is:

$$\mathcal{Z}(\mathcal{E}_9) = |\chi^9_{(0,0)} + \chi^9_{(0,9)} + \chi^9_{(9,0)} + \chi^9_{(1,4)} + \chi^9_{(4,1)} + \chi^9_{(4,4)}|^2 + 2|\chi^9_{(2,2)} + \chi^9_{(2,5)} + \chi^9_{(5,2)}|^2.$$

The presence of the factor 2 in the second term of the modular invariant indicates that the Ocneanu algebra $Oc(\mathcal{E}_9)$ is non-commutative. It is isomorphic to a direct sum of 36 one-dimensional blocks of $\mathbb{C}$ and of 9 copies of two-dimensional matrices $M_2(\mathbb{C})$, its dimension is 72. The modular subalgebra is $J = \{0_0, 1_0, 2_0\}$ and the Ocneanu algebra $Oc(\mathcal{E}_9)$ involves $\mathcal{E}_9\,\dot{\otimes}_J\,\mathcal{E}_9$ and a non-commutative matrix complement (see [27] for more details). The Ocneanu graph is made of $12 \times 6 = 72$ vertices, corresponding to three copies of the $\mathcal{E}_9$ graph and three copies of its module graph $\mathcal{E}_9/3$. The quantum mass is $m(Oc(\mathcal{E}_9)) = \frac{m(\mathcal{E}_9)\,m(\mathcal{E}_9)}{m(J)} = m(\mathcal{A}_9) = 432(7+4\sqrt{3})$, where $m(J) = 3$. Note that the quadratic sum rule can be checked ($\dim(\mathcal{B}\mathcal{E}_9) = \sum_\lambda d_\lambda^2(\mathcal{E}_9) = \sum_x d_x^2(\mathcal{E}_9) = 518\,976$) but the linear sum rule does not hold: $d_H = 4656$ but $d_V = 5448$.

6.3.4. The exceptional module of the $\mathcal{E}_9$ graph (no self-fusion)

The $\mathcal{E}_9^* = \mathcal{E}_9/3$ graph is a module over the graph algebra $\mathcal{A}_9$ and over the graph algebra $\mathcal{E}_9$. It has the same norm $\beta = [3]_q = 1+\sqrt{3}$ as $\mathcal{A}_9$ and $\mathcal{E}_9$. The $\mathcal{E}_9^*$ graph is associated with the same modular invariant as $\mathcal{E}_9$. Furthermore, the Ocneanu algebra $Oc(\mathcal{E}_9^*)$ is isomorphic to $Oc(\mathcal{E}_9)$. But the module structures of $\mathcal{E}_9^*$ over $\mathcal{A}_9$ and over $Oc(\mathcal{E}_9) \equiv Oc(\mathcal{E}_9^*)$ are not the same as for $\mathcal{E}_9$: The annular matrices F_λ and dual annular matrices S_x differ from those of $\mathcal{E}_9$. The quadratic sum rule hold and read $\dim(\mathcal{B}\mathcal{E}_9^*) = 754\,272$, but the linear sum rule does not hold: $d_H = 5\,616$ but $d_V = 6\,552$.

6.3.5. The exceptional $\mathcal{E}_{21}$ graph (self-fusion)

The $\mathcal{E}_{21}$ graph has self-fusion and possesses 24 vertices denoted as $0, 2, \ldots, 23$. The unit vertex is 0, the conjugated generators are 1 and 2. Complex conjugation corresponds to the symmetry with respect to the horizontal axis joining vertices 0 and 21 of the $\mathcal{E}_{21}$ graph given in Fig. 4. Triality is equal to the labels taken modulo 3. The norm of the $\mathcal{E}_{21}$ graph is $\beta = \frac{1}{2}(1+\sqrt{2}+\sqrt{6})$. Actually all quantum dimensions are of the kind $(a, b, c, d) = a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}$, for appropriate values of a, b, c, d. The quantum mass is $m(\mathcal{E}_{21}) = 24(18+10\sqrt{3}+\sqrt{6(97+56\sqrt{3})})$. The associated modular invariant partition function is:

$$\begin{aligned}\mathcal{Z}(\mathcal{E}_{21}) = &|\chi^{21}_{(0,0)} + \chi^{21}_{(4,4)} + \chi^{21}_{(6,6)} + \chi^{21}_{(10,10)} + \chi^{21}_{(0,21)} + \chi^{21}_{(21,0)} + \chi^{21}_{(1,10)} + \chi^{21}_{(10,1)} + \chi^{21}_{(4,13)} \\ &+ \chi^{21}_{(13,4)} + \chi^{21}_{(6,9)} + \chi^{21}_{(9,6)}|^2 + |\chi^{21}_{(0,6)} + \chi^{21}_{(6,0)} + \chi^{21}_{(0,15)} + \chi^{21}_{(15,0)} + \chi^{21}_{(4,7)} + \chi^{21}_{(7,4)} \\ &+ \chi^{21}_{(4,10)} + \chi^{21}_{(10,4)} + \chi^{21}_{(6,15)} + \chi^{21}_{(15,6)} + \chi^{21}_{(7,10)} + \chi^{21}_{(10,7)}|^2 .\end{aligned}$$

The modular subalgebra is $J = \{0, 21\}$ and a realization of the Ocneanu algebra is $Oc(\mathcal{E}_{21}) = \mathcal{E}_{21} \dot{\otimes}_J \mathcal{E}_{21}$, with the identifications $a \dot{\otimes}_J u\, b \equiv a\, u^* \dot{\otimes}_J b$, for all $u \in J$ and $a, b \in \mathcal{E}_{21}$. The Ocneanu graph involves 12 copies of $\mathcal{E}_{21}$. The dimension of $Oc(\mathcal{E}_{21})$ is 288 (see [13,47] for more details). Its quantum mass is given by $m\,[Oc(\mathcal{E}_{21})] = \frac{m[\mathcal{E}_{21}]\, m[\mathcal{E}_{21}]}{m[J]} = m\,[\mathcal{A}_{21}]$, where $m\,[J] = 2$. Numerically $m\,[Oc(\mathcal{E}_{21})] = 1728(201 + 142\sqrt{2} + 116\sqrt{3} + 82\sqrt{6})$. The linear and quadratic sum rules hold and read $d_H = d_V = 288\,576$, $\dim(\mathcal{B}\mathcal{E}_{21}) = 480\,701\,952$, respectively.

6.3.6. The twisted exceptional D^t_9 (no self-fusion)

The $\mathcal{D}^t_9$ graph is a module over the graph algebra $\mathcal{A}_9$ and over the graph algebra $\mathcal{D}_9$. It is associated with the following modular invariant partition function:

$$\begin{aligned}\mathcal{Z}(\mathcal{D}^t_9) = &|\chi^9_{(0,0)} + \chi^9_{(9,0)} + \chi^9_{(0,9)}|^2 + |\chi^9_{(3,0)} + \chi^9_{(6,3)} + \chi^9_{(0,6)}|^2 + |\chi^9_{(0,3)} + \chi^9_{(6,0)} + \chi^9_{(3,6)}|^2 \\ &+ |\chi^9_{(2,2)} + \chi^9_{(5,2)} + \chi^9_{(2,5)}|^2 + |\chi^9_{(4,4)} + \chi^9_{(4,1)} + \chi^9_{(1,4)}|^2 + 2\,|\chi^9_{(3,3)}|^2 \\ &+ \left[(\chi^9_{(1,1)} + \chi^9_{(7,1)} + \chi^9_{(1,7)})\,\overline{\chi^9_{(3,3)}} + \text{h.c.}\right].\end{aligned}$$

The graph $\mathcal{D}^t_9$ appears as a module of its own algebra of quantum symmetries (calculated from the modular splitting equation). It is a generalization of the E_7 graph[11] of the $SU(2)$ system. Its quantum mass is $m(\mathcal{D}^t_9) = 72(2 + \sqrt{3})$. $Oc(\mathcal{D}^t_9)$ is obtained via an anti-automorphism called the exceptional ambichiral twist ξ, which acts on vertices of the modular subalgebra $J = \{0_0, 2_0, 3_0, 3'_0, 4_0, 5_0, \alpha^1_0, \alpha^2_0, \alpha^3_0\}$ of $\mathcal{D}_9$ (see Fig. 4), such that $\xi\,(2_0) = \alpha^2_0$, $\xi\,(\alpha_0) = 2_0$ and $\xi\,(u) = u$ for all other $u \in J$. The Ocneanu algebra $Oc(\mathcal{D}^t_9)$ involves $\mathcal{D}_9 \dot{\otimes}_J \mathcal{D}_9$ and a non-commutative matrix complement. We identify $a \dot{\otimes}_J u\, b \equiv a\xi(u^*) \dot{\otimes}_J b$ for all $u \in J$ and $a, b \in \mathcal{D}_9$. Its dimension is 55 and the quantum mass is $m(Oc(\mathcal{D}^t_9)) = m(\mathcal{A}_9) = 432(7 + 4\sqrt{3})$. The dimension is $\dim(\mathcal{B}\mathcal{D}^t_9) = 1\,167\,355$.

6.3.7. The twisted conjugate exceptional $\mathcal{D}^{t^*}_9$ (no self-fusion)

The $\mathcal{D}^{t^*}_9$ graph is a module graph over the graph algebras $\mathcal{A}_9$, $\mathcal{D}_9$ and also $\mathcal{D}^t_9$. The modular invariant partition function associated with this graph is:

$$\begin{aligned}\mathcal{Z}(\mathcal{D}^{t^*}_9) = &|\chi^9_{(0,0)} + \chi^9_{(9,0)} + \chi^9_{(0,9)}|^2 + |\chi^9_{(2,2)} + \chi^9_{(5,2)} + \chi^9_{(2,5)}|^2 + |\chi^9_{(4,4)} + \chi^9_{(4,1)} + \chi^9_{(1,4)}|^2 \\ &+ 2\,|\chi^9_{(3,3)}|^2 + \left[(\chi^9_{(0,3)} + \chi^9_{(6,0)} + \chi^9_{(3,6)})\left(\overline{\chi^9_{(3,0)}} + \overline{\chi^9_{(6,3)}} + \overline{\chi^9_{(0,6)}}\right) + \text{h.c.}\right] \\ &+ \left[(\chi^9_{(1,1)} + \chi^9_{(7,1)} + \chi^9_{(1,7)})\,\overline{\chi^9_{(3,3)}} + \text{h.c.}\right].\end{aligned}$$

The $\mathcal{D}^{t^*}_9$ graph appears as a module of its own algebra of quantum symmetries, which is also obtained via the exceptional ambichiral twist ξ acting on vertices of $J \subset \mathcal{D}_9$. The Ocneanu algebra $Oc(\mathcal{D}^{t^*}_9)$ involves also $\mathcal{D}_9 \dot{\otimes}_J \mathcal{D}_9$ and a non-commutative matrix complement, but with the identifications $a \dot{\otimes}_J u\, b = a\xi(u) \dot{\otimes}_J b$ for all $u \in J$ and $a, b \in \mathcal{D}_9$. Its dimension is 55 and the quantum mass is $m(Oc(\mathcal{D}^{t^*}_9)) = m(Oc(\mathcal{D}^t_9)) = m(\mathcal{A}_9)$. The dimension is $\dim(\mathcal{B}\mathcal{D}^{t^*}_9) = 531\,435$.

7. Comments

Overall features of quantum groupoïds and graphs associated with higher Coxeter–Dynkin systems

For an $SU(n)$ system of graphs, one expects the following pattern. The family of $\mathcal{A}_k$ graphs is easily obtained by truncation of the Weyl chambers at level k; such $\mathcal{A}_k$ graphs involve several types of oriented lines (one for

[11] The E_7 graph should better be called $\mathcal{D}^t_{16}$.

each fundamental representation of $SU(n)$). Then one can obtain several other families by using the existence of automorphisms such as complex conjugacy (leading to the $\mathcal{A}_k^*$ series), $\mathbb{Z}_p$ symmetries (leading to the orbifold $\mathcal{D}_k[p] = \mathcal{A}_k/p$ series), or a combination of these two automorphisms (leading to the $\mathcal{D}_k^*[p]$ series). From our experience with small values of n, we expect rather different families of $\mathcal{D}$ graphs, depending on whether n is even or odd. For $SU(2)$, orbifold graphs $\mathcal{D}_k[2] = \mathcal{D}_{\frac{k}{2}+2}$ exist if $k = 0, 2 \mod 4$, and they have self-fusion whenever $k = 0 \mod 4$. For $SU(3)$, orbifold graphs $\mathcal{D}_k[3]$ exist for all k, and they have self-fusion whenever $k = 0 \mod 3$. For $SU(4)$, and according to [40], we have orbifold graphs of type $\mathcal{D}_k[2]$ for all k, and they have self-fusion whenever $k = 0 \mod 2$, but we have also orbifold graphs of type $\mathcal{D}_k[4]$ for $k = 0, 2, 6 \mod 8$, and they have self-fusion whenever $k = 0 \mod 8$.

For $\mathcal{A}_k$ and $\mathcal{A}_k^*$ series, the algebra of quantum symmetries can be determined from the tensor square of the graph algebra $\mathcal{A}_k$, suitably quotiented. When $\mathcal{D}_k$ does not have self-fusion, its algebra of quantum symmetries can also be determined from the tensor square of the graph algebra $\mathcal{A}_k$, suitably quotiented with the help of appropriate generalizations of the Gannon twist. This is also the case for its corresponding conjugated series. When $\mathcal{D}_k$ graph has self-fusion, its algebra of quantum symmetries (which, in this case, is non-commutative) can be obtained as a cross-product of the graph algebra of $\mathcal{D}_k$ by the cyclic group $\mathbb{Z}_p$; this is also the case for the corresponding conjugate series. In all of these cases, the associated modular invariant is easy to obtain from the $\mathcal{A}$ modular invariant at the same level.

For a given system, it seems that one can always find a (unique) exceptional graph $\mathcal{D}^t$, without self-fusion, whose algebra of quantum symmetries is equal to the quotient of the tensor square of a particular $\mathcal{D}$ graph by an exceptional automorphism (this generalizes the (E_7, D_{10}) situation of the $SU(2)$ family). The graph $\mathcal{D}^t$ itself is then recognized as a module over its algebra of quantum symmetries. Determination of this automorphism can be performed by looking at the values of the modular operator T on vertices of the corresponding $\mathcal{A}$ graph and the induction–restriction rules from $\mathcal{A}$ to $\mathcal{D}$ [13]. The same discussion holds for the corresponding conjugated graph $\mathcal{D}^{t^*}$.

We are then left with the other exceptional graphs. They may admit self-fusion or not. When they do not, they are orbifolds of those exceptionals that enjoy self-fusion. Graphs with self-fusion are called "quantum subgroups" by A. Ocneanu, the others being only "quantum modules". Those exceptional subgroups are $E_6 \equiv \mathcal{E}_{10}$ and $E_8 \equiv \mathcal{E}_{28}$ for the $SU(2)$ system, $\mathcal{E}_5$, $\mathcal{E}_9$ and $\mathcal{E}_{21}$ for the $SU(3)$ system and $\mathcal{E}_4$, $\mathcal{E}_6$ and $\mathcal{E}_8$ for the $SU(4)$ system. Their algebra of quantum symmetries may be commutative or not. Non-commutativity can be deduced, either from the presence of integer entries bigger than 1 in the modular invariant, or from the existence of non-trivial classical symmetries in the graph itself (see footnote in Section 5.4). When the algebra of quantum symmetries $Oc(G)$ is commutative, like for E_6 and E_8 in the $SU(2)$ system, or like for $\mathcal{E}_5$, $\mathcal{E}_{21}$ in the $SU(3)$ system, it is easy to obtain the corresponding toric matrices and the $Oc(G)$ itself without having to solve the modular splitting equation, because, in these cases, one obtains $Oc(G)$ as a tensor square of G itself over the modular subalgebra J which can be determined by using the properties of the modular generator T under restriction–induction (see [13]). Of course, it is always advisable to check that the result obtained satisfies the modular splitting equation. If, however, the algebra of quantum symmetries of this exceptional graph with self-fusion is non-commutative (like for the $\mathcal{E}_9$ case), the determination of $Oc(G)$ becomes quite involved and the only method we can think of is again to use the modular splitting technique.

Once the exceptional graphs with self-fusion are known, it is not too difficult to obtain the exceptional modules: They are quotients or orbifolds of the former and often appear as particular subspaces of $Oc(G)$.

Finally, let us mention that when the graph G is a priori known, and whenever the vertex x of $Oc(G)$ can be written as $a \dot{\otimes} b$, with $a, b \in G$, it is usually possible to obtain (or recover) the toric matrices W_{x0} from the annular or essential matrices, see for instance [12] or [47]. This method, first presented in [9], is particularly easy to implement when one considers generalizations of the exceptional graphs with self-fusion E_6 and E_8 (i.e., $\mathcal{E}_5$ and $\mathcal{E}_{21}$ for the $SU(3)$ system), since $Oc(G) = G \dot{\otimes}_J G$, in those cases. One obtains $W_{x0} = \sum_{c \in J} (F_\lambda)_{ac} (F_\lambda)_{bc} = E_a.((E_b)^{\mathrm{red}})^T$, where the reduced essential matrices E_b^{red} are obtained from the E_b by keeping the matrix elements of those columns corresponding to the modular subalgebra J and putting all other entries to zero.

Graphs from modular invariants

One possibility is to rely on a given classification of the modular invariants. Such a classification exists for $SU(2)$ [7] and $SU(3)$ [22] but is not available for $SU(n)$ when $n > 3$. However there are arguments showing that the level of exceptionals cannot be too high [41], so that it is enough to explore a sizeable list of possibilities. Once a modular invariant is known, one can use the modular splitting technique and find the algebra $Oc(G)$. Generically, the Ocneanu graph involves one or several copies of the graph G itself and of its modules; this may not be so in special cases, for instance the D_{odd} case of the $SU(2)$ system or in the conjugated series of the $SU(3)$ system, but then, other techniques of determination can be used (cf the above discussion). Once the graph G is obtained, one has still to check that the result obtained gives rise to a "good" theory of representations (here $SU(3)$); otherwise, it should be discarded. We believe that the precise meaning of this sentence is that the graph obtained should give rise to a Kuperberg spider [30]; another possibility is to use the existence of a self-connection, as defined by A. Ocneanu in [40]. As already mentioned, we believe that the two notions coincide but it is clear that some more work is needed in this direction. The list of graphs expected to provide an answer to the $SU(4)$ classification problem is given in [40].

Conformal embeddings

Another possibility leading to interesting candidates for graphs G of higher Coxeter–Dynkin systems is to use the existence of conformal embeddings of affine algebras — a subject that we did not touch in this paper. One should be aware that the (1) List of modular invariants, (2) List of conformal embeddings, (3) List of graphs belonging to higher Coxeter–Dynkin systems (or defining Ocneanu quantum groupoïds) are distinct problems.

It happens that, for $SU(2)$ and $SU(3)$, all exceptional graphs with self-fusion correspond to particular conformal embeddings, but other such embeddings lead to orbifolds or to members (with small level) of the $\mathcal{D}$ series. In the case of $SU(4)$, it seems that there is one exceptional graph with self-fusion not associated with any conformal embedding.

Conformal embeddings of affine algebras at level k of the type $\widehat{su}(n)_k \subset \hat{g}_1$, where g is a simple Lie algebra, simply laced or not, can be associated with graphs that are candidates for becoming members, at level k, of the Coxeter–Dynkin system of $SU(n)$. The condition of being conformal imposes equality of the central charges:

$$\frac{(n^2-1)k}{k+n} = \frac{\dim(g)}{1+\kappa(g)} \tag{29}$$

where $\dim(g)$ is the dimension of g and $\kappa(g)$ its dual Coxeter number. This equation is easy to solve for all $SU(n)$ systems. In the case $n = 2$ there are three non-trivial solutions: $E_6(\equiv\mathcal{E}_{10})$, for $g = B_2 = \mathrm{spin}(5)$, then $E_8(\equiv\mathcal{E}_{28})$ for $g = G_2$ and finally $D_4(\equiv\mathcal{D}_4)$, for $g = A_2 = su(3)$. In the case $n = 3$ there are many more solutions; let us just mention those that give rise to exceptional graphs with self-fusion: $\mathcal{E}_5$ for $g = A_5 = su(6)$, then $\mathcal{E}_9$ for $g = A_6 = su(7)$ and finally $\mathcal{E}_{21}$ for $g = E_7$.

Other generalizations

The algebra of quantum symmetries described in the previous section refers to quantum groupoïds for which a basis of matrix units, for the vertical product, is made of double triangles of type $GGAGG$, where G is any graph of the system (A-type, D-type, exceptional type etc.). However one may replace these double triangles by others, of type $GGKGG$, whenever G is a K module. This was apparently not studied.

About the definitions of $Oc(G)$

The most pleasant definition of $Oc(G)$ is to take it as the algebra of characters (or irreps) for the horizontal product on $\widehat{\mathcal{B}G}$. This amounts to considering the center of $\widehat{\mathcal{B}G}$ (for the horizontal multiplication $\widehat{\circ}$) and analysing its structure when endowed with a product inherited from the vertical multiplication on $\mathcal{B}G$. However, determining it in this way requires a priori the calculation of several sets (finite but huge) of generalized $6J$ symbols. It seems that nobody ever did it this way (the family of $6J$ symbols is not even known for the exceptional cases of the $SU(2)$ system!) Rather, the generators O_x were obtained as explained in step II of the modular splitting technique. A clear discussion relating these two types of concepts would be welcome.

Frontiers

The possibility of associating higher order algebraic systems (somehow generalizing universal enveloping algebras and their root systems) with graphs that are members of higher Coxeter–Dynkin families is certainly a fascinating perspective, which was not discussed in this paper.

Conclusion

The quantum groupoïd aspects of these systems are still largely under-studied. As already stated previously, and in agreement with popular wisdom, every graph G belonging to an $SU(n)$ system should give rise, and conversely, to an “Ocneanu quantum groupoïd”. Altogether these objects constitute a particular family of finite dimensional weak Hopf algebras. However, many general properties still need clarification and every single particular diagram deserves more study — for instance the explicit determination of the different types of cells (generalized $6J$ symbols), is an open problem.

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